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Projection Operators for $\eta\pi$ systems —Version I—

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abstract

This note is based on one of my preprints, BNL-QGS-96-31, in which the significance of the moments $H(30)$ and $H(32)$ was discussed. The present note is an amplified version of my previous one, with emphasis on the projection operators.

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1 Introduction

In one of my preprints, BNL-QGS-96-31[1], I discussed the significance of the moments $H(30)$ and $H(32)$ as well as $H(40)$ and $H(42)$ and argued that there must exist a P_+ state close to the $a_2(1320)$ region.

In that note, I gave a series of combinations of the moments—projection operators—which could shed light on the D_+ and P_+ states. (A prime example of the projection operators is the $H(LM)$ as defined in the next section.) In the present note, I expand the notion of the projection operators, to derive some more useful ones which retain unnatural-parity waves only, eliminating natural-parity waves. Since these projection operators are based on unnormalized moment technique, the resulting expressions for unnatural-parity waves are, by definition, unique and do *not* suffer the ambiguity problem inherent in the $\eta\pi$ systems.

In our publication of the exotic meson[2], the emphasis was given to the mass dependence of the natural-parity waves, D_+ and P_+, in order to assess the significance of the 1^{-+} exotic state in our data. The present note is intended for a study of the unnatural-parity waves in which the ambiguity problem inherent in the $\eta\pi$ systems is absent.

2 Unnormalized Moments

The unnormalized moments have the following general expressions[3]:

$$H(LM) = \sum_i d^L_{M0}(\theta_i)\, \cos M\phi_i \tag{1}$$

summing over all events in a given mass bin. $H(LM)$'s of interest are given by

$$H(00) = \sum 1 \tag{2a}$$

$$H(10) = \sum \cos\theta_i \tag{2b}$$

$$H(11) = -\frac{1}{\sqrt{2}} \sum \sin\theta_i\, \cos\phi_i \tag{2c}$$

$$H(20) = \frac{1}{2} \sum (3\cos^2\theta_i - 1) \tag{2d}$$

$$H(21) = -\sqrt{\frac{3}{2}} \sum \sin\theta_i\, \cos\theta_i\, \cos\phi_i \tag{2e}$$

$$H(22) = \frac{\sqrt{6}}{4} \sum \sin^2\theta_i\, \cos(2\phi_i) \tag{2f}$$

$$H(30) = \frac{1}{2} \sum (5\cos^3\theta_i - 3\cos\theta_i) \tag{2g}$$

$$H(31) = -\frac{\sqrt{3}}{4} \sum \sin\theta_i \left(5\cos^2\theta_i - 1\right) \cos\phi_i \tag{2h}$$

$$H(32) = \sqrt{\frac{15}{8}} \sum \sin^2\theta_i \, \cos\theta_i \, \cos(2\phi_i) \tag{2i}$$

$$H(40) = \frac{1}{8} \sum \left(35\cos^4\theta_i - 30\cos^2\theta_i + 3\right) \tag{2j}$$

$$H(41) = -\frac{\sqrt{5}}{4} \sum \sin\theta_i \left(7\cos^3\theta_i - 3\cos\theta_i\right) \cos\phi_i \tag{2k}$$

$$H(42) = \sqrt{\frac{5}{32}} \sum \sin^2\theta_i \left(7\cos^2\theta_i - 1\right) \cos(2\phi_i) \tag{2l}$$

where the summation extends over all observed events in a given $\eta\pi^-$ mass bin. It is noted that the $H(30)$ and $H(32)$ moments are odd in $z = \cos\theta$ whereas $H(40)$ and $H(42)$ are even in z, as expected.

Recall that the unnormalized angular distribution is given by[3]

$$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi}\right) H(LM) D^{L\,*}_{M0}(\phi, \theta, 0) \tag{3}$$

where

$$H(LM) = \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} V_{\ell m} V^*_{\ell' m'} (\ell' m' L M | \ell m)(\ell' 0 L 0 | \ell 0) \tag{4}$$

and $V_{\ell m}$'s are the partial waves to be determined. It is seen that, if $L = 4$, then one must have $\ell = \ell' = 2$. Since the $a_2(1320)$ is produced mainly in the D_+ wave, one needs to consider only $m = \pm 1$ and $m' = \pm 1$. This in turn leads to the conclusion that one must have $M = 0$ or $M = 2$. If $L = 3$, then one sees that, for example, $\ell = 2$ and $\ell' = 1$, and the only wave which can interfere with D_+ is the P_+. This shows that one must have again $m = \pm 1$ and $m' = \pm 1$ and, similarly, $M = 0$ or $M = 2$. Note that the same conclusion can be reached from Tables Ia and Ib of Ref[3].

It is helpful to write down explicitly the contributions of each wave. From the aforementioned tables, one finds

$$H(00) = S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2 \tag{5a}$$

$$H(10) = \frac{1}{\sqrt{3}} S_0 P_0 + \frac{2}{\sqrt{15}} P_0 D_0 + \frac{1}{\sqrt{5}} (P_- D_- + P_+ D_+) \tag{5b}$$

$$H(11) = \frac{1}{\sqrt{6}} S_0 P_- + \frac{1}{\sqrt{10}} P_0 D_- - \frac{1}{\sqrt{30}} P_- D_0 \tag{5c}$$

$$H(20) = \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{5} P_0^2 - \frac{1}{5}(P_-^2 + P_+^2) + \frac{2}{7} D_0^2 + \frac{1}{7}(D_-^2 + D_+^2) \tag{5d}$$

$$H(21) = \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{5}\sqrt{\frac{3}{2}} P_0 P_- + \frac{1}{7\sqrt{2}} D_0 D_- \tag{5e}$$

$$H(22) = \frac{1}{5}\sqrt{\frac{3}{2}}(P_-^2 - P_+^2) + \frac{1}{7}\sqrt{\frac{3}{2}}(D_-^2 - D_+^2) \tag{5f}$$

$$H(30) = \frac{3}{7\sqrt{5}}(\sqrt{3} P_0 D_0 - P_- D_- - P_+ D_+) \tag{5g}$$

$$H(31) = \frac{1}{7}\sqrt{\frac{3}{5}}(2 P_0 D_- + \sqrt{3} P_- D_0) \tag{5h}$$

$$H(32) = \frac{1}{7}\sqrt{\frac{3}{2}}(P_- D_- - P_+ D_+) \tag{5i}$$

$$H(40) = \frac{2}{7} D_0^2 - \frac{4}{21}(D_-^2 + D_+^2) \tag{5j}$$

$$H(41) = \frac{1}{7}\sqrt{\frac{5}{3}} D_0 D_- \tag{5k}$$

$$H(42) = \frac{\sqrt{10}}{21}(D_-^2 - D_+^2) \tag{5l}$$

where A^2 stands for $|A|^2$ and AB for $2\mathrm{Re}\{AB^*\}$. Note that the moments $H(LM)$'s with $M = 1$ have contributions from the partial waves with unnatural-parity exchange only.

One can write down certain projection operators derived from $H(LM)$'s. From the formulas listed above, one sees that

$$P_2^{(+)} = \sqrt{3}\, H(20) + \sqrt{2}\, H(22) \tag{6a}$$

$$= \sqrt{\frac{3}{5}}\, S_0 D_0 + \frac{2\sqrt{3}}{5}(P_0^2 - P_+^2) + \frac{2\sqrt{3}}{7}(D_0^2 + D_-^2) \tag{6b}$$

$$P_2^{(-)} = \sqrt{3}\, H(20) - \sqrt{2}\, H(22) \tag{6c}$$

$$= \sqrt{\frac{3}{5}}\, S_0 D_0 + \frac{2\sqrt{3}}{5}(P_0^2 - P_-^2) + \frac{2\sqrt{3}}{7}(D_0^2 + D_+^2) \tag{6d}$$

$$P_3^{(+)} = \sqrt{5}\, H(30) + \sqrt{6}\, H(32) \;=\; \frac{3\sqrt{3}}{7} P_0 D_0 - \frac{6}{7} P_+ D_+ \tag{6e}$$

$$P_3^{(-)} = \sqrt{5}\,H(30) - \sqrt{6}\,H(32) \;=\; \frac{3\sqrt{3}}{7}P_0D_0 - \frac{6}{7}P_-D_- \tag{6f}$$

$$P_4^{(+)} = \sqrt{5}\,H(40) + 2\sqrt{2}\,H(42) \;=\; \frac{2\sqrt{5}}{7}D_0^2 - \frac{8\sqrt{5}}{21}D_+^2 \tag{6g}$$

$$P_4^{(-)} = \sqrt{5}\,H(40) - 2\sqrt{2}\,H(42) \;=\; \frac{2\sqrt{5}}{7}D_0^2 - \frac{8\sqrt{5}}{21}D_-^2 \tag{6h}$$

where A^2 stands for $|A|^2$ and AB for $2\mathrm{Re}\{AB^*\}$. It is noted that the projection operators $P_2^{(+)}$, $P_3^{(-)}$ and $P_4^{(-)}$ are devoid of the main wave D_+. Therefore, they are naturally more sensitive to the secondary waves in the problem.

Now glance at these moments as posted by Dave Thompson in our secure area. The moments $H(40)$ and $H(42)$ show a BW shape, as expected for the resonant D_+ wave. The crucial observation is that the unnormalized moments $H(30)$ and $H(32)$ also show a BW shape. One can understand better the meaning of this, by noting that a BW could be written

$$e^{i\delta}\,\sin\delta \tag{7}$$

where the phase δ is a rapidly varying function of the mass m, as shown below

$$\cot\delta = \frac{m_0^2 - m^2}{m_0\Gamma_0} \tag{8}$$

Here m_0 and Γ_0 are the mass and width of the $a_2(1320)$. If $P_+ = a\exp[\alpha]$ and both a and α are independent of m, the interference term

$$\cos(\alpha - \delta)\,\sin\delta$$

must go through zeroes whenever

$$\alpha = \delta + n\frac{\pi}{2}$$

and n is odd. The form of the interference term therefore should be very different in general from the BW shape, which is given simply by $\sin^2\delta$. It is clear now that the P_+ wave must also be resonant with approximately the same mass and width of the $a_2(1320)$, since the shapes of $H(30)$ and $H(32)$ are almost identical to those of $H(40)$ and $H(42)$.

3 Additional Projection Operators

From ($5a$), ($5f$) and ($5l$), one can eliminate natural-parity waves via

$$
\begin{aligned}
Q_1 &= H(00) + 5\sqrt{\frac{2}{3}}\,H(22) + 3\sqrt{\frac{2}{5}}\,H(42) \\
&= S_0^2 + P_0^2 + 2P_-^2 + D_0^2 + 2D_-^2
\end{aligned}
\tag{9}
$$

Note that this is not quite the sum of all the unnatural-parity waves, since the waves $|P_-|^2$ and $|D_-|^2$ appear twice. The importance of this result is that the particular combination of the unnatural-parity waves above does *not* suffer from ambiguities inherent in the system. Thus the projection operator Q_1 provides a unique insight into the behavior these waves in the $a_2(1320)$ region. One can form a different combination of unnatural-parity waves using the same technique

$$
\begin{aligned}
Q_2 &= H(00) + 5\sqrt{\frac{2}{3}}\,H(22) + \frac{3}{2}\,H(40) \\
&= S_0^2 + P_0^2 + 2P_-^2 + \frac{10}{7}(D_0^2 + D_-^2)
\end{aligned}
\tag{10}
$$

The projection operator Q_2 again gives rise to an independent and unique combination of unnatural-parity waves.

One can obtain yet another projection operator which has the same effect as Q_1 above but is indenpendent. In order to obtain such an expression, one has first to write down the angular distribution explicitly. From eqs. (32), (33) and (34) of Ref.[3], one can write, for the angular distribution,

$$
\begin{aligned}
I(\Omega) &= \left(\frac{1}{4\pi}\right) \left| S_0 + \sqrt{3} P_0 \, d^{\,1}_{00} + \sqrt{5} D_0 \, d^{\,2}_{00} + (\sqrt{6} P_- \, d^{\,1}_{10} + \sqrt{10} D_- \, d^{\,2}_{10}) \cos\phi \right|^2 \\
&\quad + \left(\frac{1}{4\pi}\right) \left| \sqrt{6} P_+ \, d^{\,1}_{10} + \sqrt{10} D_+ \, d^{\,2}_{10} \right|^2 \sin^2\phi \\
&= \left(\frac{1}{4\pi}\right) \Big| S_0 + \sqrt{3} P_0 \, \cos\theta + \frac{\sqrt{5}}{2} D_0 \, (3\cos^2\theta - 1) \\
&\qquad\qquad - \sqrt{3}(P_- + \sqrt{5} D_- \, \cos\theta) \sin\theta \, \cos\phi \Big|^2 \\
&\quad + \left(\frac{3}{4\pi}\right) \left| P_+ + \sqrt{5} D_+ \, \cos\theta \right|^2 \sin^2\theta \, \sin^2\phi
\end{aligned}
\tag{11}
$$

where the θ dependence of the small d-functions has been suppressed. Using the well-known relationship

$$
\int \mathrm{d}\cos\theta \; d^{\,\ell}_{mm'}(\theta) \, d^{\,\ell'}_{mm'}(\theta) = \left(\frac{2}{2\ell+1}\right) \delta_{\ell\ell'}
$$

it is simple to show that

$$
\begin{aligned}
Q_3 &= \int \mathrm{d}\Omega \; I(\Omega) \; (1 + 2\cos 2\phi) \\
&= S_0^2 + P_0^2 + 2P_-^2 + D_0^2 + 2D_-^2
\end{aligned}
\tag{12}
$$

It should be noted that this projection operator does not involve the small d-functions and hence it is fundamentally different from Q_1. In analogy to $H(LM)$'s, the unnormalized moment Q_3 is given by

$$
Q_3 = \sum_i^n (1 + 2\cos 2\phi_i) \tag{13}
$$

for a given mass bin containing n events.

Let $Q_0 = H(00)$ for convenience of notation, i.e.

$$
Q_0 = S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2 \tag{14}
$$

Comparing this with Q_3, one sees that

$$Q_4 = Q_3 - Q_0 = 2 \int \mathrm{d}\Omega \ I(\Omega) \ \cos 2\phi$$
$$= P_-^2 + D_-^2 - P_+^2 - D_+^2 \tag{15}$$

This shows that, in the limit of vanishing P_- and D_-, the projection operator Q_4 is given by the sum of$-P_+^2$ and $-D_+^2$. An alternative expression for the projection operator above is given by

$$Q_5 = 5\sqrt{\frac{2}{3}}\,H(22) + 3\sqrt{\frac{2}{5}}\,H(42) = P_-^2 + D_-^2 - P_+^2 - D_+^2 \tag{16}$$

Note that this combination moments has already appeared in Q_1. Another useful operator isolates out P waves without S or D waves; observe that

$$Q_6 = 5\sqrt{\frac{2}{3}}\,H(22) - 3\sqrt{\frac{5}{2}}\,H(42) = P_-^2 - P_+^2 \tag{17}$$

Acknowledgments

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References

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