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Formulas for Partial-Wave Analysis Version II

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ABSTRACT

A concise description is given of the formulas used in partial-wave analyses of $K\bar{K}\pi$ systems.

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1. Introduction

We describe in this note all the relevant formulas which are used in the BNL program for the partial-wave analysis of the $K\bar{K}\pi$ system. The formalism is based on the so-called isobar model, which in the current version involves three isobars; two $K^*(890)$ states in the $K\pi$ and the $\bar{K}\pi$ channels and the $a_0/\delta(980)$ state in the $K\bar{K}$ channel.

Most of the related background material can be found in a series of notes and reports written by the author [1,2,3] and, in addition, in a paper written by the author in collaboration with Trueman[4]. The reader may also consult Hansen et al.[5] and Ascoli et al.[6] for earlier work on the isobar formalism; however, their formulas are somewhat different from those given in this note.

Section 2 exhibits the Lorentz-invariant phase-space element, along with the various Lorentz frames relevant for specifying partial waves. Section 3 deals with the problem of defining the partial waves in the isobar model; existence of a complete orthonormal set of functions corresponding to the isobar intermediate states is worth noting. Modifications to these functions in the reflectivity basis are given in Section 4, as well as the parameterization of the spin-density matrix in the same basis. The task of defining G- and C-eigenstates are handled in Section 5. Section 6 covers the coupled-channel Breit-Wigner form necessary for describing the $a_0/\delta(980)$. Section 7 takes up the problem of constructing suitable likelihood functions.

2. Phase-space Considerations

For concreteness we start with the following reaction for production of a resonance X^0:

$$\pi^- p \to \; X^0 n, \quad X^0 \to \; K\bar{K}\pi \tag{2.1}$$

The Lorentz-invariant phase-space element for this process is given by, neglecting factors which depend only on $\sqrt{s}$ and W,[7]

$$d\rho \propto \; d\Omega^* \;\; (p \; d\Omega \; dW) \;\; (q \; d\Omega_h \; dw) \tag{2.2}$$

where $\Omega^* = (\theta^*, \phi^*)$ represents the polar and azimuthal angles of the resonance X^0 of mass W in the overall center-of-mass (CM) system; $\Omega = (\Theta, \Phi)$ denotes the angles describing the orientation of the momentum $\vec{p}$ of the isobar with mass w in the $K\bar{K}\pi$ rest frame (RF); $\Omega_h = (\theta, \phi)$ denotes the angles corresponding to the momentum $\vec{q}$ of one of the decay products of the isobar in its helicity RF.

Let s_1 and s_2 be the squares of the effective masses describing the Dalitz plot, with $s_1 = w^2$. Then, for a fixed w, one has

$$ds_2 \propto \frac{W}{w}\, pq\; d\cos\theta \tag{2.3}$$

so that

$$d\rho \propto\; d\Omega^*\; dR\; ds_1\; ds_2 \tag{2.4}$$

where $R = (\Phi, \Theta, \phi)$ represents the Euler angles,

$$dR \;\propto\; d\Phi\; d\cos\Theta\; d\phi \tag{2.5}$$

We choose the angles $\Omega = (\Theta, \Phi)$ to be the polar and azimuthal angles describing the isobar orientation $\vec{p}$ in the Jackson frame (X^0RF), i.e. the z-axis is chosen to be parallel to the beam momentum in the X^0RF and the y-axis is along the production normal $\vec{beam} \times \vec{X}^0$, defined in the overall CM. The angles $\Omega_h = (\theta, \phi)$ specify the orientation $\vec{q}$ of one of the decay products of the isobar in the helicity frame (isobar RF), in which the axes are defined such that $\hat{z}_h = \hat{p}$ and $\hat{y}_h \propto\; \hat{z} \times\; \hat{p}$ and $\hat{z}$ is the z-axis in the Jackson frame('$\hat{}$' signifies a unit vector). Note that one has $\hat{X}^0 \;\Rightarrow\; (\theta^*, \phi^*)$ in the overall CM, $\hat{p} \;\Rightarrow\; (\Theta, \Phi)$ in the X^0RF and $\hat{q} \;\Rightarrow\; (\theta, \phi)$ in the isobar-RF. In practice, the reaction does not depend on the ϕ^* variable, so it can be integrated over, and the element $d\cos\theta^*$ can be related to dt where t is the four-momentum transfer from the initial π^- to the final X^0. The 'Lorentz factor' pq which appears in the phase-space formula, Eq. (2.2), gives the 'correct' weighting factor for the reaction; as such it it not a part of the 'invariant' amplitude we consider in the next section.

A full description of the three-body system requires five variables; we have chosen the four angles $\Omega = (\Theta, \Phi)$ and $\Omega_h = (\theta, \phi)$, and the isobar mass w as these variables. We

show in the next section a complete set of orthonormal functions spanning the four angles, with the result that the mass/momentum dependent factors enter only through w.

The differential cross section for the reaction Eq. (2.1) can now be written, suppressing the factors which depend only on $\sqrt{s}$ and the W,

$$\frac{d\sigma}{dt\ dW\ d\tau} \propto\ |\mathcal{M}|^2 pq \tag{2.6}$$

where the variables relevant for partial-wave analyses are collectively denoted by τ,

$$\begin{aligned} \tau &= \{\Omega, \Omega_h, w\} \\ d\tau &= d\Omega\ d\Omega_h\ dw \\ &= d\Phi\ d\cos\Theta\ d\phi\ d\cos\theta\ dw \end{aligned} \tag{2.7}$$

The purpose of the partial-wave analysis is to parameterize the invariant amplitude $\mathcal{M}$ as a function of τ and analyze the data for each bin with given t and W. Thus, the function one needs to calculate for the purpose is the 'distribution function',

$$I(\tau) = |\mathcal{M}|^2 \tag{2.8}$$

Another way to write down the differential cross section is use the Dalitz-plot variables $\{s_1, s_2\}$:

$$\frac{d\sigma}{dt\ dW\ dR\ ds_1\ ds_2} \propto\ I(\tau) \tag{2.9}$$

In this form, the phase-space factors have disappeared from the formula and it is symmetric under interchange of subscripts 1 and 2 and, formally, independent of a specific choice of an isobar. The Euler angles R can now be defined merely to 'fix' the orientation of the 3-body system in its own RF. This formula shows that, for each point in a grid of (t, W), the Monte-Carlo events can be simply generated via a set of five random numbers specifying R, s_1 and s_2, with the 'Lorentz factor' = 1.

3. Decay Amplitudes

We shall use 'a'and 'b' to denote the following set of quantum numbers needed to describe a $K\bar{K}\pi$ system:

$$a = \{\ell s J^P(v)\}, \quad b = \{am\} \tag{3.1}$$

where 'v' stands for the isobar and s is its spin, and ℓ is the orbital angular momentum between the isobar and the bachelor particle, and J is the total spin with m as its z-component and P is the intrinsic parity given by $P = (-1)^{\ell+s+1}$. Then the amplitude for the X^0 decay into the isobar, followed in turn by its decay, is given by[8]

$$A_b(\tau) \propto \tilde{J}\tilde{s} \sum_{\lambda} D^{J}_{m\lambda}{}^{*}(\Phi,\Theta,0)\ D^{s}_{\lambda 0}{}^{*}(\phi,\theta,0)\ f_{a\lambda}(w) \tag{3.2}$$

where

$$\tilde{J} = \sqrt{(2J+1)}, \qquad \tilde{s} = \sqrt{(2s+1)} \tag{3.3}$$

s and λ are the spin and helicity of the isobar, and $f_{a\lambda}(w)$ is the helicity coupling 'constant' (but it depends on the w). It is related to the ℓs-coupling constant g_a through a standard prescription[9],

$$f_{a\lambda}(w) \propto \frac{\tilde{\ell}}{\tilde{J}}(\ell 0 s\lambda|J\lambda)\ Q^{v}_{\ell s}(w)\ g_a \tag{3.4}$$

In the isobar model, g_a is considered independent of w, but it can depend on the W. In partial-wave analyses, g_a is in fact absorbed into the production amplitudes, which are then the parameters to be determined in maximum likelihood fits.

The factor 'Q' contains all the 'known' dependence on the isobar mass w:

$$Q^{v}_{\ell s}(w) = F_\ell(p)\, F_s(q)\, \Delta_v(w) \tag{3.5}$$

where the functions 'F' denote the angular-momentum barrier factors and are defined below. The Δ-function has the standard Breit-Wigner form for the isobar v,

$$\Delta_v(w) = \frac{w_0\Gamma_0}{w_0^2 - w^2 - iw_0\Gamma_v(w)} \tag{3.6}$$

with

$$\Gamma_v(w) = \Gamma_0 \frac{w_0}{w} \frac{q}{q_0} \frac{F_s^2(q)}{F_s^2(q_0)} \tag{3.7}$$

where w_0 and Γ_0 are the mass and the width of the isobar and $q_0 = q(w_0)$ so that $\Gamma_v(w_0) = \Gamma_0$ and $\Delta_v(w_0) = 1$. Note that the w dependence of the width is assumed to be given by the two-body phase-space factor q/w with the decay amplitude given by $F_s(q)$.

From the form of $F_s(q)$ given below, the energy dependence of the width is[10]

$$\Gamma_v(w) \propto \frac{q^{2s+1}}{w} \tag{3.8}$$

Since the width formula in terms of the Lorentz-invariant decay amplitude has a w^{-2} dependence, the above formula implies that the invariant amplitude is proportional to $\sqrt{w}F_s(q)$. One way to understand this dependence is to consider an elastic scattering in the two-body channel in which the isobar v is formed. The Lorentz-invariant scattering amplitude in a partial wave s is[11], suppressing numerical constants,

$$T_s \propto \frac{w}{q}(2s+1) P_s(x) e^{i\delta_s} \sin\delta_s \tag{3.9}$$

where the partial-wave S-matrix has been given the usual unitary form $\exp[2i\delta_s]$ in terms of the phase shift δ_s, and $P_s(x)$ is the usual Legendre polynomial with the argument x representing cosine of the scattering angle. A resonance in a partial wave s can be written in a conventional form

$$\cot\delta_s = \frac{w_0^2 - w^2}{w_0 \Gamma_v(w)} \tag{3.10}$$

so that

$$T_s \propto \frac{w}{q}(2s+1) P_s(x) \frac{w_0 \Gamma_v(w)}{w_0^2 - w^2 - i w_0 \Gamma_v(w)} \tag{3.11}$$

It is seen that the w dependence as given in Eq. (3.8) makes the invariant scattering amplitude T_s to have the simplest possible singularity-free q and x dependence in the numerator, i.e. a polynomial of degree s in the variable $(\vec{q}_i \cdot \vec{q}_f)$ only, where $\vec{q}_i$ and $\vec{q}_f$ are

the initial and final breakup momenta of magnitude q in the CM system for the elastic scattering.

The functions $F_\ell(p)$ and $F_s(q)$ are the Blatt-Weisskopf centrifugal-barrier factors as given by von Hippel and Quigg[12],

$$F_0\left(p\right) = 1 \tag{3.12}$$

$$F_1\left(p\right) = \sqrt{\frac{2z}{z+1}} \tag{3.13}$$

$$F_2\left(p\right) = \sqrt{\frac{13z^2}{\left(z-3\right)^2+9z}} \tag{3.14}$$

$$F_3\left(p\right) = \sqrt{\frac{277z^3}{z\left(z-15\right)^2+9\left(2z-5\right)^2}} \tag{3.15}$$

$$F_4\left(p\right) = \sqrt{\frac{12746z^4}{\left(z^2-45z+105\right)^2+25z\left(2z-21\right)^2}} \tag{3.16}$$

where $z = (p/p_R)^2$ and $p_R = 0.1973$ (GeV/c) corresponding to 1 fermi. Normally, the barrier factor is defined such that $F_\ell(p) \Rightarrow 1$ as $p \Rightarrow \infty$, but it here is renormalized so that $F_\ell(p) = 1$ for $z = 1$, in order to have the fitted parameter for the ℓ wave reflect its 'actual' size at $p = p_R$. Note that

$$A_b \propto p^\ell \qquad \text{for } p \approx 0 \tag{3.17}$$

and

$$A_b \propto q^s \qquad \text{for } q \approx 0 \tag{3.18}$$

In other words, we demand that the amplitude A_b has the 'correct' p and q dependence near threshold.

Combining Eq. (3.2) and Eq. (3.4), one may separate out explicitly the w dependence from that of the angles and write, dropping the constant g_a,

$$A_b\left(\tau\right) = E_m^{J\ell s\,*}\left(\Omega,\Omega_h\right)\; Q_{\ell s}^v\left(w\right) \tag{3.19}$$

where the functions 'E', which has no explicit dependence on the isobar v, form a complete orthonormal set in the space spanned by the four angles Ω and Ω_h:

$$E_m^{J\ell s\,*}\left(\Omega,\Omega_h\right) = \tilde{J}\tilde{s}\sum_\lambda D_{m\lambda}^{J\,*}\left(\Phi,\Theta,\phi\right)\; d_{\lambda 0}^s\left(\theta\right)\; \left[\frac{\tilde{\ell}}{\tilde{J}}\left(\ell 0 s\lambda|J\lambda\right)\right] \tag{3.20}$$

Note that

$$E_m^{J\ell s}(\Omega,\Omega_h)=1, \qquad \text{if } J=\ell=s=m=0 \tag{3.21}$$

and also that

$$\sum_\lambda \left[\frac{\tilde{\ell}}{\tilde{J}}\ (\ell 0 s\lambda|J\lambda)\right]^2 = 1 \tag{3.22}$$

It can be shown that

$$\int d\Omega\ d\Omega_h\ E_m^{J\ell s\ *}(\Omega,\Omega_h)\ E_{m'}^{J'\ell's'}(\Omega,\Omega_h) = (4\pi)^2\ \delta_{JJ'}\ \delta_{\ell\ell'}\ \delta_{ss'}\ \delta_{mm'} \tag{3.23}$$

Formally, the functions 'E' are related to the 'ket' states $|\ell sJm\rangle$ and $|\Omega,\Omega_h\rangle$ via

$$\langle\Omega,\Omega_h|\ell sJm\rangle = E_m^{J\ell s\ *}(\Omega,\Omega_h) \tag{3.24}$$

with the 'ket' states normalized according to

$$\begin{aligned}\langle\ell sJm|\ell's'J'm'\rangle &= \delta_{\ell\ell'}\ \delta_{ss'}\ \delta_{JJ'}\ \delta_{mm'}\\ \langle\Omega,\Omega_h|\Omega',\Omega_h'\rangle &= (4\pi)^2\ \delta\left(\Omega-\Omega'\right)\ \delta\left(\Omega_h-\Omega_h'\right)\end{aligned} \tag{3.25}$$

It is seen that the 'E' functions have 'rational' normalizations in the sense that the 'ket' states $|\Omega\rangle$ and $|\Omega_h\rangle$ are each normalized to its full 4π steradians.

4. Reflectivity Basis

The parity conservation in the production process Eq. (2.1) is taken into account through change of basis via a reflection operator[4] defined in the overall CM system, to bring the density matrix into a block-diagonal form. The new basis involves introduction of the reflectivity ϵ through

$$|\epsilon am\rangle = \left[|am\rangle - \epsilon P\,(-1)^{J-m}\,|a-m\rangle\right]\theta\,(m) \tag{4.1}$$

where P is the parity of the state 'a' and

$$\begin{aligned}\theta\,(m) &= \frac{1}{\sqrt{2}}, \qquad m>0\\ &= \frac{1}{2}, \qquad m=0\\ &= 0, \qquad m<0\end{aligned} \tag{4.2}$$

The reflectivity ϵ is here defined such that it coincides with the naturality of exchanged Regge trajectories. Note that

$$|\epsilon a m\rangle = 0 \text{ for } m = 0, \qquad \text{if} \quad \epsilon = P\,(-1)^J \tag{4.3}$$

In the new basis, the 'E' functions become

$${}^{\epsilon}E^{J\ell s}_{m}{}^{*}\,(\Omega,\Omega_h) = \tilde{\ell}\tilde{s}\sum_{\lambda}{}^{\epsilon}D^{JP}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi)\;d^{s}_{\lambda 0}\,(\theta)\;\;(\ell 0 s\lambda|J\lambda) \tag{4.4}$$

where

$${}^{\epsilon}D^{JP}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) = \theta\,(m)\left[D^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) - \epsilon P\,(-1)^{J-m}\,D^{J}_{-m\lambda}{}^{*}\,(\Phi,\Theta,\phi)\right] \tag{4.5}$$

Since $P = (-1)^{\ell+s+1}$, the 'E' functions can be recast into

$${}^{\epsilon}E^{J\ell s}_{m}{}^{*}\,(\Omega,\Omega_h) = \tilde{\ell}\tilde{s}\sum_{\lambda}{}^{\epsilon}B^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi)\;d^{s}_{\lambda 0}\,(\theta)\;\;(\ell 0 s\lambda|J\lambda) \tag{4.6}$$

where

$${}^{\epsilon}B^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) = \theta\,(m)\left[D^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) + \epsilon D^{J}_{m\lambda}\,(\Phi,\Theta,\phi)\right] \tag{4.7}$$

It is readily ascertained that

$$\begin{aligned}{}^{\epsilon}B^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) &= 2\theta\,(m)\,\mathrm{Re}\,D^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) \quad (\epsilon = +1)\\ &= 2i\theta\,(m)\,\mathrm{Im}\,D^{J}_{m\lambda}{}^{*}\,(\Phi,\Theta,\phi) \quad (\epsilon = -1)\end{aligned} \tag{4.8}$$

In other words, the functions 'B' are purely real for $\epsilon = +1$ and purely imaginary for $\epsilon = -1$. The normalization for the functions 'E' in the new basis is given by

$$\int d\Omega\;d\Omega_h\;{}^{\epsilon}E^{J\ell s}_{m}{}^{*}\,(\Omega,\Omega_h)\;\;{}^{\epsilon'}E^{J'\ell's'}_{m'}\,(\Omega,\Omega_h) = (4\pi)^2\,\delta_{\epsilon\epsilon'}\delta_{JJ'}\;\delta_{\ell\ell'}\;\delta_{ss'}\;\delta_{mm'} \tag{4.9}$$

It should be borne in mind that the functions 'B' defined above, in contrast to the 'E' and the 'D' functions, do not satisfy the orthonormal relationships.

We exhibit finally the decay amplitudes in the new basis:

$${}^{\epsilon}A_b\,(\tau) = \;{}^{\epsilon}E^{J\ell s}_{m}{}^{*}\,(\Omega,\Omega_h)\;\;Q^{v}_{\ell s}\,(w) \tag{4.10}$$

It is in fact in this form that the decay amplitudes have been coded into the BNL partial-wave analysis program.

The parameters 'V' of the fit are those that enter into the definition of the spin-density matrix:

$$ {}^{\epsilon}\rho_{bb'} = \sum_{k}^{K} {}^{\epsilon}V_{bk}\ {}^{\epsilon}V^{*}_{b'k} \tag{4.11} $$

where the K is the rank of the density matrix ($K = 2$ in our case). Note that, owing to parity conservation in the production process, the density matrix becomes block-diagonal in the sense that ϵ with a prime does not appear its definition. The 'V' is given in the triangular representation[4],

$$ \begin{aligned} {}^{\epsilon}V_{bk} &= 0, \qquad k > n_{\epsilon}(b) \\ &= \text{real}, \qquad k = n_{\epsilon}(b) \\ &= \text{complex}, \qquad \text{otherwise} \end{aligned} \tag{4.12} $$

where $n_{\epsilon}(b)$ is an ordinal number ranging from 1 to $N_{\epsilon}(b)$ which is the total number of different states b for a given ϵ. Now the square of the invariant amplitude $\mathcal{M}$ assumes the form

$$ I(\tau) = \sum_{\epsilon bb'} {}^{\epsilon}\rho_{bb'}\ {}^{\epsilon}A_{b}(\tau)\ {}^{\epsilon}A^{*}_{b'}(\tau) \tag{4.13} $$

Or, introducing a function,

$$ {}^{\epsilon}U_{k}(\tau) = \sum_{b} {}^{\epsilon}V_{bk}\ {}^{\epsilon}A_{b}(\tau) \tag{4.14} $$

the $I(\tau)$ can be succinctly written

$$ I(\tau) = \sum_{\epsilon k} |{}^{\epsilon}U_{k}(\tau)|^{2} \tag{4.15} $$

indicating that the distribution function $I(\tau)$ is in fact a sum of $2K$ non-interfering terms, each of which results from a sum of fully interfering states $|\epsilon a m\rangle$.

5. G- and C-parity Eigenstates

A detailed description of the problem of constructing the eigenstates under the G- and the C-parity operations can be found in a note written by the author[3]. Here only the bare minimum necessary for exposition of relevant formulas are given.

Let τ be a given configuration of the $K\bar{K}\pi$ system. We denote by $\bar{\tau}$ that obtained by interchanging K and $\bar{K}$. Suppressing indices ϵ, a and m, which are irrelevant for this discussion, one may form

$$\psi_{\pm}(\tau) = \frac{1}{\sqrt{2}}[A(\tau) \pm A(\bar{\tau})] \tag{5.1}$$

Suppose now that the decay amplitude A is expanded as a function of orbital angular momenta L for the $K\bar{K}$ system in its own RF. Its G-parity is $(-1)^{L+1}$ if the $K\bar{K}$ system is charged; while its C-parity is $(-1)^L$ if it is neutral. Note in particular that this argument holds regardless of the nature of the isobars involved—this is a general result.

It is clear now that for a <u>neutral</u> $K\bar{K}\pi$ system,

$$G\psi_{\pm} = \pm\psi_{\pm} \tag{5.2}$$

for the state containing a charged π, while

$$C\psi_{\pm} = \pm\psi_{\pm} \tag{5.3}$$

for the one containing a neutral π. It should be stressed that these results are independent of the isospin of the neutral $K\bar{K}\pi$ system. Thus, isospin considerations are irrelevant for a neutral $K\bar{K}\pi$ system, as long as appropriate G- or C- parity eigenstates are used in the analysis.

Let μ be the eigenvalue for either the G- or C-parity operation. Then, the eigenstates are written, fully restoring all the indices,

$${}^{\epsilon}\psi_{b\mu}(\tau) = \frac{1}{\sqrt{2}}[\ {}^{\epsilon}A_b(\tau) + \mu\ {}^{\epsilon}A_b(\bar{\tau})] \tag{5.4}$$

where $\mu = \pm 1$. The specification of decay amplitudes for the $K\bar{K}\pi$ system is now complete. For completeness, we exhibit the distribution functions 'I' in terms of these eigenstates. We introduce a shorthand notation $c = \{b, \mu\}$, to write

$$I(\tau) = \sum_{\epsilon cc'} {}^{\epsilon}\rho_{cc'}\ {}^{\epsilon}\psi_c(\tau)\ {}^{\epsilon}\psi^*_{c'}(\tau) \tag{5.5}$$

The 'U' functions now take the form

$${}^{\epsilon}U_k(\tau) = \sum_c {}^{\epsilon}V_{ck}\ {}^{\epsilon}\psi_c(\tau) \tag{5.6}$$

and the $I(\tau)$ is again given by

$$I(\tau) = \sum_{\epsilon k} |{}^{\epsilon}U_k(\tau)|^2 \tag{5.7}$$

6. Parametrization for the $a_0(980)$ Isobar

We use for the Breit-Wigner form for the $a_0/\delta(980)$ the coupled-channel formula given by Flatté[13], with one modification: the w dependence of the width is assumed to be given by the Lorentz-invariant phase-space factor q/w, not merely by q as used by Flatté.

The δ is seen to decay into two channels, $\pi\eta$ and $K\bar{K}$, and they will be labelled 1 and 2 in this note. Then, the Breit-Wigner formula replacing Eq. (3.6) can be written

$$\Delta_\delta(w) = \frac{|D_\delta(w_t)|}{D_\delta(w)} \tag{6.1}$$

where

$$D_\delta(w) = w_\delta^2 - w^2 - iw_\delta \frac{w_t}{w}\left(g_1^2 q_1 + g_2^2 q_2\right) \tag{6.2}$$

The parameters g_1 and g_2 are the dimensionless coupling constants and q_1 and q_2 are the breakup momenta for the $\pi\eta$ and $K\bar{K}$ channels, respectively. The constant w_t is the $K\bar{K}$ threshold, so that the magnitude of the Δ-function is normalized to one at the threshold.

Let w_0 and Γ_0 be the mass and the width as observed in the $\pi\eta$ channel. Then, by demanding that Eq. (6.1) approximates these parameters

$$\Gamma_\delta(w) \to\ w^2 - w_0^2 - iw_0\Gamma_0, \qquad \text{as} \quad w \to\ w_0 \tag{6.3}$$

with $q_2 = ik_2$, where k_2 is the magnitude of q_2 below the $K\bar{K}$ threshold, one obtains

$$w_\delta^2 = w_0^2 - r\frac{k_2\,(w_0)}{q_1\,(w_0)}w_0\Gamma_0 \tag{6.4}$$

and

$$g_1^2 = \frac{w_0^2\Gamma_0}{w_t w_\delta q_1\,(w_0)} \tag{6.5}$$

Here one has set $r = g_2^2/g_1^2$.

The shape of the resonance peak as given by Eq. (6.1) is highly asymmetric near w_t; so in practice one has to increase Γ_0 or decrease r or both to approximate a simple Breit-Wigner form of a given mass w_0 and width Γ_0. For further comments on this point, the reader is referred to Ref. 14.

7. Maximum-likelihood Methods

In this section we exhibit the form of the likelihood function used in our partial-wave analysis, along with the normalization integrals for all Monte-Carlo events and also for accepted MC events. We show in addition the formula for calculating the 'predicted' or 'acceptance-corrected' number of events.

The likelihood function for finding 'n' events of a given bin with a finite acceptance $\eta(\tau)$ is defined as a product of the probabilities,

$$\mathcal{L} \propto \left[\frac{\bar{n}^n}{n!}\,\mathrm{e}^{-\bar{n}}\right]\,\prod_i^n\,\left[\frac{I\,(\tau_i)}{\int I\,(\tau)\,\eta\,(\tau)\;pq\;d\tau}\right] \tag{7.1}$$

where the first bracket is the Poisson probability for 'n' events. This is the so-called extended likelihood function, in the sense that the Poisson distribution for 'n' itself is included in the likelihood function[15]. Note that the expectation value $\bar{n}$ for n is given by

$$\bar{n} \propto \int I\,(\tau)\,\eta\,(\tau)\;pq\;d\tau \tag{7.2}$$

The likelihood function $\mathcal{L}$ can now be written, dropping the factors depending on n alone,

$$\mathcal{L} \propto \left[\prod_i^n\;I\,(\tau_i)\right]\exp\left[-\int I\,(\tau)\,\eta\,(\tau)\;pq\;d\tau\right]$$

The 'log' of the likelihood function now has the form,

$$\ln \mathcal{L} \propto \sum_{i}^{n} \ln I(\tau_i) - \int I(\tau)\,\eta(\tau)\; pq\; d\tau \tag{7.3}$$

We shall adopt the following shorthand notation for indices:

$$\begin{aligned} \alpha &= \{\epsilon k c\} = \{\epsilon k a \mu m\} \\ \alpha' &= \{\epsilon k c'\} = \{\epsilon k a' \mu' m'\} \end{aligned} \tag{7.4}$$

where μ stands for either G- or C-parity. Note that 'primes' do not appear for ϵ and k; they apply to non- interfering indices. Now the distribution functions $I(\tau)$ assume the following compact form,

$$I(\tau) = \sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, \psi_\alpha(\tau)\; \psi^*_{\alpha'}(\tau) \tag{7.5}$$

where ψ is the symmetrized decay amplitude 'A' as defined previously. As such, ψ does not depend on the index k.

The normalization integrals are most expeditiously obtained through the Monte-Carlo events. Let M be the number of MC events generated, and let M_x be the number accepted by the finite geometry of the experiment and other software cuts. The MC acceptance is then given by $\eta_x = M_x/M$. One needs two sets of normalizations,

$$\Psi_{\alpha\alpha'} = \frac{1}{M} \sum_{i}^{M} \psi_\alpha(\tau_i)\; \psi^*_{\alpha'}(\tau_i) \tag{7.6}$$

for the full MC sample and

$$\Psi^x_{\alpha\alpha'} = \frac{1}{M_x} \sum_{i}^{M_x} \psi_\alpha(\tau_i)\; \psi^*_{\alpha'}(\tau_i) \tag{7.7}$$

for the accepted MC sample, so that

$$\eta_x \Psi^x_{\alpha\alpha'} = \frac{1}{M} \sum_{i}^{M_x} \psi_\alpha(\tau_i)\; \psi^*_{\alpha'}(\tau_i) \tag{7.8}$$

is the true accepted normalization integral, obtained by replacing M_x in the denominator with M in Eq. (7.7).

The 'log' of the extended likelihood function $\mathcal{L}$ can therefore be written

$$\ln \mathcal{L} = \sum_{i}^{n} \ln \left[\sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \psi_\alpha(\tau_i) \, \psi_{\alpha'}^*(\tau_i) \right] - \eta_x \left[\sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \Psi_{\alpha\alpha'}^x \right] \tag{7.9}$$

where n is the number of experimental events in a given bin. With an extended likelihood function, the parameters themselves now have an absolute normalization. It can be shown that, by substituting 'V's by 'cV's where c is a constant independent of α, and then by differentiating $\ln \mathcal{L}$ by c^2, that the 'V's are normalized according to

$$\eta_x \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \Psi_{\alpha\alpha'}^x = n \tag{7.10}$$

(The partial derivative should equal to zero at the minimum.)
The 'predicted' number of events from the fit is

$$N = \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \Psi_{\alpha\alpha'} \tag{7.11}$$

One may renormalize the parameter via

$$V \to \sqrt{\frac{n}{\eta_x}} \, V \tag{7.12}$$

Then, in terms of the new parameters, the 'log' of the likelihood function reads

$$\ln \mathcal{L} = \sum_{i}^{n} \ln \left[\sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \psi_\alpha(\tau_i) \, \psi_{\alpha'}^*(\tau_i) \right] - n \left[\sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \Psi_{\alpha\alpha'}^x \right] \tag{7.13}$$

and the normalization condition assumes the form

$$\sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \Psi_{\alpha\alpha'}^x = 1 \tag{7.14}$$

With this normalization, the fitted 'V's are, to first order, independent of variations among different bins, and a set of fitted parameters in a given bin is, therefore, an excellent starting value for the neighboring bins. The predicted number of events is now given by, in terms of the new parameters,

$$N = \frac{n}{\eta_x} \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \, \Psi_{\alpha\alpha'} \tag{7.15}$$

Note that $\eta_x = M_x/M$ would be the predicted fraction of accepted events if the data were distributed isotropically as in MC events. This formula shows that the 'acceptance-corrected' event number N is proportional to the observed event n divided by η_x— the summation involving the fitted parameters 'V's is precisely what is needed to 'correct' for the anisotropy inherent in the experimental data.

Consider next a case in which two separate triggers, 1 and 2, result in the same event-types under study. The joint likelihood is simply the product of the two likelihoods $\mathcal{L}_1$ and $\mathcal{L}_2$. And the 'log' of the likelihood is given by

$$\ln \mathcal{L} = \sum_i^{n_1+n_2} \ln \left[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \psi_\alpha(\tau_i) \, \psi^*_{\alpha'}(\tau_i) \right] - \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \left[\eta_1 \Psi^{(1)}_{\alpha\alpha'} + \eta_2 \Psi^{(2)}_{\alpha\alpha'} \right] \tag{7.16}$$

where the subscripts and superscripts 1 and 2 stand for the triggers 1 and 2. The normalizations are found to be, using the techniques as before,

$$\eta_1 \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi^{(1)}_{\alpha\alpha'} = n_1 \tag{7.17}$$

$$\eta_2 \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi^{(2)}_{\alpha\alpha'} = n_2 \tag{7.18}$$

and the predicted number of events is again given by Eq. (7.11).

One may redefine the parameters in a manner similar to Eq. (7.12), as follows:

$$V \to \sqrt{\frac{n_1+n_2}{\eta_1+\eta_2}} \, V \tag{7.19}$$

Then the new 'log' of likelihood is given by

$$\begin{aligned} \ln \mathcal{L} = & \sum_i^{n_1+n_2} \ln \left[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \psi_\alpha(\tau_i) \, \psi^*_{\alpha'}(\tau_i) \right] \\ & - (n_1+n_2) \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \left[\frac{\eta_1}{\eta_1+\eta_2} \Psi^{(1)}_{\alpha\alpha'} + \frac{\eta_2}{\eta_1+\eta_2} \Psi^{(2)}_{\alpha\alpha'} \right] \end{aligned} \tag{7.20}$$

and the new normalizations are

$$\left(\frac{n_1+n_2}{\eta_1+\eta_2} \right) \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi^{(1)}_{\alpha\alpha'} = \frac{n_1}{\eta_1} \tag{7.21}$$

$$\left(\frac{n_1+n_2}{\eta_1+\eta_2} \right) \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi^{(2)}_{\alpha\alpha'} = \frac{n_2}{\eta_2} \tag{7.22}$$

The predicted number of events assume the form, in terms of the new parameters,

$$N = \left(\frac{n_1+n_2}{\eta_1+\eta_2} \right) \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi_{\alpha\alpha'} \tag{7.23}$$

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