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Formulas for Partial-Wave Analysis —Addendum I—

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abstract

A prescription is given for applying the method of extended maximum-likelihood analyses to two different data sets.

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1 Introduction

We describe in this note the method of extended maximum-likelihood analyses applied to two different data sets.

In a previous note[1], we dealt with the case of two different triggers which give rise to the same event types of interest. There it was assumed that the two triggers are taken simultaneously for a given number of hours in an experiment. In this note, we are concerned instead with a situation in which the same or different trigger has been used in two different time periods, yielding two different data sets for the same event type of interest. Even if the same trigger setup had been used, the hardware conditions never remain the same and so the experimental acceptance must be different for the two data sets.

The task is to devise a set of formulas to be used when the two data sets are combined in the extended maximum-likelihood analysis, in which one given set of physics amplitudes is to be determined, corrected for acceptance.

2 Extended Maximum-likelihood Methods

We shall adopt the same notations as in the previous note[1] wherever possible. The likelihood function for finding 'n' events of a given bin with a finite acceptance $\eta(\tau)$ is defined as a product of the probabilities,

$$\mathcal{L} \propto \left[\frac{\bar{n}^n}{n!}\,\mathrm{e}^{-\bar{n}}\right] \prod_i^n \left[\frac{I(\tau_i)\,\eta(\tau_i)\,f(\tau_i)}{\int I(\tau)\,\eta(\tau)\;\mathrm{d}\rho(\tau)}\right] \tag{1}$$

where the first bracket is the Poisson probability for 'n' events; $I(\tau)$ is the distribution function; and $\mathrm{d}\rho(\tau) = f(\tau)\,\mathrm{d}\tau$. The Lorentz-invariant phase-space element is embodied in $\rho(\tau)$ (see Appendix).

The expectation value $\bar{n}$ for n is given by

$$\bar{n} = \int I(\tau)\,\eta(\tau)\;\mathrm{d}\rho(\tau) \tag{2}$$

so that

$$N = \int I(\tau)\;\mathrm{d}\rho(\tau) \tag{3}$$

where N is the predicted number of events, corrected for acceptance. So $\bar{n}$ is equal to N if the acceptance is 100%, i.e. $\eta(\tau) = 1$. Dropping the factors independent of V's, the 'log' of the likelihood function now assume the form,

$$\ln \mathcal{L} = \sum_{i}^{n} \ln I(\tau_i) \; - \int I(\tau)\,\eta(\tau)\;\mathrm{d}\rho(\tau) \tag{4}$$

Now the distribution functions $I(\tau)$ can be written

$$I(\tau) = \sum_{\alpha\alpha'} V_\alpha \; V^*_{\alpha'} \; \psi_\alpha(\tau) \; \psi^*_{\alpha'}(\tau) \tag{5}$$

where ψ_α is a suitable decay amplitude with the quantum numbers specified by α. Note that α and α' contain in general certain indices (or quantum numbers) that remain the same, e.g. the reflectivity ϵ and—if necessary—the baryon helicities k which may be required in a typical reaction under study. The complex variables V_α are the parameters to be fitted in the maximum likelihood method. The experimental acceptance is denoted by $\eta(\tau)$ and it modifies the normalization integrals. In general, $\eta(\tau)$ is not only a function of τ but also of other unspecified variables such as the momentum transfer t; it is assumed that the data set has already been suitably devided into bins of a given set in these varaviables, e.g. into different t bins.

We now follow the same technique employed in the previous note. Assume that a suitable set $\{V_\alpha\}$ has been found in a maximum likelihood method. Then, by substituting V_α by cV'_α where c is a constant independent of α, one must have

$$\frac{\partial \ln \mathcal{L}}{\partial c^2} = 0 \tag{6}$$

From this we see that the 'V's are normalized according to

$$n = \sum_{\alpha\alpha'} V_\alpha \; V^*_{\alpha'} \int \psi_\alpha(\tau) \; \psi^*_{\alpha'}(\tau)\,\eta(\tau) \;\mathrm{d}\rho(\tau) \tag{7}$$

so that

$$N = \sum_{\alpha\alpha'} V_\alpha \; V^*_{\alpha'} \int \psi_\alpha(\tau) \; \psi^*_{\alpha'}(\tau) \;\mathrm{d}\rho(\tau) \tag{8}$$

An inspection of (2) and (7) reveals that the best estimate for $\bar{n}$ is in fact n itself—a natural result. Note the self consistency in the formulation of the extended likelihood method, in the sense that (3) and (8) are identical.

The normalization integrals are most expeditiously obtained through the Monte-Carlo events. Let M_1 be the number of MC events generated for the data set 1, and let m_1 be the number accepted by the acceptance η_1 embodying the finite geometry of the experiment and other software cuts. And similary for the data set 2. The MC acceptance are then given by $\eta_1 = m_1/M_1$ and $\eta_2 = m_2/M_2$. By definition, the accepted m_j event set is a subset of the full MC sample with M_j events, where $j = 1, 2$. The appropriate normalization integrals are

$$\begin{aligned} \Psi^{(1)}_{\alpha\alpha'} &= \frac{1}{M_1} \sum_i^{M_1} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \\ \Psi^{(2)}_{\alpha\alpha'} &= \frac{1}{M_2} \sum_i^{M_2} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \end{aligned} \tag{9}$$

for the full MC samples and

$$\begin{aligned} \Phi^{(1)}_{\alpha\alpha'} &= \frac{1}{m_1} \sum_i^{m_1} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \\ \Phi^{(2)}_{\alpha\alpha'} &= \frac{1}{m_2} \sum_i^{m_2} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \end{aligned} \tag{10}$$

for the accepted MC samples, so that

$$\begin{aligned} \eta_1 \Phi^{(1)}_{\alpha\alpha'} &= \frac{1}{M_1} \sum_i^{m_1} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \\ \eta_1 \Phi^{(2)}_{\alpha\alpha'} &= \frac{1}{M_2} \sum_i^{m_2} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \end{aligned} \tag{11}$$

are the true accepted normalization integrals, obtained by replacing m_j in the denominator with M_j in (10). The MC samples $\{M_1\}$ and $\{M_2\}$ refer to the same event types. So in the limit of large MC events, one must have

$$\Psi_{\alpha\alpha'} = \Psi^{(1)}_{\alpha\alpha'} = \Psi^{(2)}_{\alpha\alpha'} \tag{12}$$

The joint likelihood for the two data sets 1 and 2 is simply the product of the two likelihoods $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\eta_1(\tau)$ and $\eta_2(\tau)$ be the corresponding acceptances. The 'log' of the joint likelihood is therefore given by

$$\ln \mathcal{L} = \sum_i^{n_1+n_2} \ln I(\tau_i) \ - \int I(\tau)\ [\frac{N_1}{N}\,\eta_1(\tau) + \frac{N_2}{N}\,\eta_2(\tau)]\ \mathrm{d}\rho(\tau) \tag{13}$$

where n_1 (n_2) is the number of experimental events for the sample 1 (2) and N_1 (N_2) is the predicted number of events for the sample 1 (2). The key ingredient incorporated in the formula above[a] is that the parameter sets $\{V\}$ for samples 1 and 2 have the normalizations N_1 and N_2 in (8) and that each must be renormalized by $\sqrt{N_1/N}$ and $\sqrt{N_2/N}$, in order that a common set $\{V\}$ can be used in the minimization process. (See a later paragraph for a further clarification of this point.) One sees that

$$\ln\mathcal{L} = \sum_i^{n_1+n_2} \ln\Big[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \psi_\alpha(\tau_i)\psi^*_{\alpha'}(\tau_i)\Big] - \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Big[\frac{N_1}{N}\,\eta_1\Phi^{(1)}_{\alpha\alpha'} + \frac{N_2}{N}\,\eta_2\Phi^{(2)}_{\alpha\alpha'}\Big] \tag{14}$$

and

$$\frac{N_1}{N}\,\eta_1\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Phi^{(1)}_{\alpha\alpha'} + \frac{N_2}{N}\,\eta_2\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Phi^{(2)}_{\alpha\alpha'} = n_1 + n_2 \tag{15}$$

and the predicted numbers of events N are

$$\frac{N_1}{N}\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi^{(1)}_{\alpha\alpha'} + \frac{N_2}{N}\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi^{(2)}_{\alpha\alpha'} = N_1 + N_2 = N \tag{16}$$

or, from (12),

$$\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi_{\alpha\alpha'} = N \tag{17}$$

For an acceptable fit, one must demand, for the samples 1 and 2 seaparately,

$$\begin{aligned}\frac{N_1}{N}\,\eta_1\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Phi^{(1)}_{\alpha\alpha'} &\simeq n_1\\ \frac{N_2}{N}\,\eta_2\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Phi^{(2)}_{\alpha\alpha'} &\simeq n_2\end{aligned} \tag{18}$$

There is one serious problem in the formulation of the likelihood analysis outlined above; the numbers N_1 and N_2 are unknown quantities. One obvious remedy is that one sets

$$N_1 = n_1/\eta_1, \qquad \text{and} \qquad N_2 = n_2/\eta_2 \tag{19}$$

initially; minimize (14); and then one substitutes the resulting parameter set $\{V\}$ into (17) and (18) to solve for new N_1 and N_2. It is hoped that a few iteration would yield stable N_1 and N_2. Note that the initial values of N_1 and N_2 given in (19) lead to nonsensical results in (18). Note, in addition, that both (18) and (19) break down if either η_1 or η_2

[a] The reader may note that this formula is fundamentally different from the one given in my previous note. Here we deal with two different data sets; in my previous note, I dealt with two different triggers obtained in a single experimental run.

(correspondingly, either n_1 or n_2) is zero. One must remember in this case that the whole idea of combining two data samples 1 and 2 becomes moot; note in particular that (13) itself becomes meaningless since either N_1 or N_2 cannot be determined and hence N itself becomes indeterminate.

There is an alternative—perhaps better—way of determining N_1 and N_2. One simply performs the extended likelihood analysis on both data sets 1 and 2 separately, and use the resultant values N_1, N_2 and N in a combined fit. To the extent that the N's thus determined are already close to the correct values, one needs to perform only a few additional iterations for convergence in the combined fit. Note that the initial V's to be introduced in the combined fit are given by

$$V_\alpha = \frac{1}{2}\left(\sqrt{\frac{N}{N_1}}\,V_\alpha^{(1)} + \sqrt{\frac{N}{N_1}}\,V_\alpha^{(2)}\right) \tag{20}$$

where the superscripts refer to the V's found in separately in data sets 1 and 2. The new V's should have a normalization equal to (17).

One may redefine the parameters in a manner similar to the previous note,

$$V \to \sqrt{\frac{n_1+n_2}{\eta_1+\eta_2}}\,V \tag{21}$$

so that V's are nearly independent of the number of events from bin to bin. Then the new 'log' of likelihood is given by

$$\begin{aligned}\ln\mathcal{L} = \sum_i^{n_1+n_2}\ \ln\Big[\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \psi_\alpha(\tau_i)\psi_{\alpha'}^*(\tau_i)\Big] \\ -\left(\frac{n_1+n_2}{\eta_1+\eta_2}\right)\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \Big[\frac{N_1}{N}\,\eta_1\Phi_{\alpha\alpha'}^{(1)} + \frac{N_2}{N}\,\eta_2\Phi_{\alpha\alpha'}^{(2)}\Big]\end{aligned} \tag{22}$$

and the new normalizations are

$$\frac{N_1}{N}\,\eta_1\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \Phi_{\alpha\alpha'}^{(1)} + \frac{N_2}{N}\,\eta_2\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \Phi_{\alpha\alpha'}^{(2)} = \eta_1+\eta_2 \tag{23}$$

and

$$\left(\frac{n_1+n_2}{\eta_1+\eta_2}\right)\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \Psi_{\alpha\alpha'} = N \tag{24}$$

One sees that, from (23),

$$\begin{aligned}\frac{N_1}{N}\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \Phi_{\alpha\alpha'}^{(1)} \simeq 1 \\ \frac{N_2}{N}\sum_{\alpha\alpha'}\ V_\alpha V_{\alpha'}^*\ \Phi_{\alpha\alpha'}^{(2)} \simeq 1\end{aligned} \tag{25}$$

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Appendix

Consider a reaction, for example,

$$\pi^- p \to X^- p, \qquad X^- \to K_s K^- \pi^0 \tag{26}$$

Then, the phase-space element is given by

$$f(\tau) = p_f, \qquad \mathrm{d}\tau = (\mathrm{d}\alpha\, \mathrm{d}\cos\beta\, \mathrm{d}\gamma)\ (\mathrm{d}E_1\, \mathrm{d}E_2) \tag{27}$$

where p_f is the breakup momentum of X and the final proton in the overall center-of-mass system; $\{\alpha, \beta, \gamma\}$ are the Euler angles describing the orientation of the $K\bar{K}\pi$ system in its rest frame; and E_1 and E_2 are the energies of any two of the three particles in the same rest frame.The reader may consult Appendix B of my CERN Yellow Report[2] for further information.

Let W be the effective mass of the $K\bar{K}\pi$ system. The form of the phase-space element (27) implies that the cross section must be given by, neglecting the factors which depend on $\sqrt{s}$ alone,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta\, \mathrm{d}W^2} \propto \int |\mathcal{M}(\tau)|^2\, \mathrm{d}\rho(\tau) = \int |\mathcal{M}(\tau)|^2\, f(\tau)\, \mathrm{d}\tau \tag{28}$$

where $\mathcal{M}(\tau)$ is the Lorentz-invariant amplitude for Reaction (26) and θ is the angle between p_f and the beam direction in the overall center-of-mass system. So the distribution function $I(\tau)$ introduced in (1) has two forms:

$$I(\tau) \propto |\mathcal{M}(\tau)|^2 \tag{29}$$

if the binning is done in W^2 and $\cos\theta$, and

$$I(\tau) \propto W\, |\mathcal{M}(\tau)|^2 \tag{30}$$

if the binning is in W and $\cos\theta$.

Let t stand for the four-momentum transfer between the initial and final protons in Reaction (26). Note that, neglecting the factors which depend on $\sqrt{s}$ alone and so independent of W, one has

$$\mathrm{d}t \propto p_f \, \mathrm{d}\cos\theta \tag{31}$$

so that

$$I(\tau)\, f(\tau) \propto |\mathcal{M}(\tau)|^2 \tag{32}$$

if the binning is done in W^2 and t, and

$$I(\tau)\, f(\tau) \propto W\, |\mathcal{M}(\tau)|^2 \tag{33}$$

if the binning is in W and t.

Reaction (26) comes with two 'external' variables t and W. If the analysis is carried out in two dimensions t and W^2, the formula (32) shows that the phase-space element is constant (independent of W or t) and hence can be set to 1, if its variables are chosen to be $\{\alpha, \beta, \gamma\}$ and $\{E_1, E_2\}$. To summarize, one sees that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t\,\mathrm{d}W^2} \propto \left(\frac{1}{p_f}\right) \int |\mathcal{M}(\tau)|^2\, \mathrm{d}\rho(\tau) = \int |\mathcal{M}(\tau)|^2\, (\mathrm{d}\alpha\, \mathrm{d}\cos\beta\, \mathrm{d}\gamma)\ (\mathrm{d}E_1\, \mathrm{d}E_2) \tag{34}$$

It is very important that one keep scrupulous track of mass-dependence in a global fit to a finite region of mass W. Only with the choice of the variables chosen above, one can set the Lorentz-invariant phase-space factor to a constant—to be mated to a set of Lorentz-invariant production and decay amplitudes in $\mathcal{M}(\tau)$.

References

[1] S. U. Chung, "Formulas for partial-Wave Analysis—Version II," BNL preprint BNL-QGS-93-05.

[2] S. U. Chung, "Spin Formalisms," CERN Yellow Report CERN 71-8 (1971).