The BNL Four-Body Partial Wave Analysis System

Based on BNL-QGS-93-05 (S.U. Chung)

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Abstract

We have modified the BNL three-body partial wave program developed by S.U. Chung to study the four-body meson system in the reaction $\pi^- p \to K^+ \bar{K}^0 \pi^- \pi^- p$. In this note a detailed description is given of the partial wave analysis of $K^+ \bar{K}^0 \pi^- \pi^-$ system.

1 Partial Wave Analysis Formalism

In this note all the relevant formulas which were used in the partial wave analysis (PWA) of the $K^+\bar{K}^0\pi^-\pi^-$ system [1] are described. We had employed the BNL version of the PWA formalism in analysis of a previous experiment E771 [2] and modified it for the four body final system.

Isobar Model

The PWA formalism used in the analysis was based on the isobar model, i.e., it was assumed that three of the four bodies (and two of the three bodies) interacted with the remaining particle acting as an independent spectator. The isobar model has been used extensively by various experiments to describe their data on three body final states [2, 3, 4, 5]. With four mesons in the final state, the PWA algebra becomes somewhat more complicated but the ideas are more or less the same as The state formed by the $K^+\bar{K}^0\pi^-\pi^-$ system was considered to decay by three sequential two-body (isobar and bachelor particles) decays. The anticipated decay chains were:

$$
\begin{array}{l}
\pi^- p \;\rightarrow\; \underbrace{X^-}\, p \\
\qquad\qquad \hookrightarrow \underbrace{f_1(1285)/\eta(1295)}\,\pi^- \\
\qquad\qquad\qquad\qquad \hookrightarrow \underbrace{a_0(980)}\,\pi^- \\
\qquad\qquad\qquad\qquad\qquad \hookrightarrow \underbrace{\bar{K}^0}\,K^+ \\
\qquad\qquad\qquad\qquad\qquad\quad \hookrightarrow \pi^+\pi^-
\end{array}
$$

In this analysis involved two isobars for the primary decay; $f_1(1285)$ and $\eta(1295)$ states in the $K^+\bar{K}^0\pi^-$ channel, and an isobar for the secondary decay; $a_0(980)$ state [1] in the $K^+\bar{K}^0$ channel.

<u>Description of the $K^+\bar{K}^0\pi^-\pi^-$ States</u>

The $K^+\bar{K}^0\pi^-\pi^-$ system is described as a superposition of states characterized by the following set of quantum numbers:

$$a = \{\ell s \ell' \sigma J^{PG}\ (w, w')\} \tag{1.1}$$

$$b = \{am\} \tag{1.2}$$

where

- ℓ is the orbital angular momentum between the bachelor particle (π^-) and the isobar $[f_1(1285)/\eta(1295)]$ for the primary decay,
- s is the spin of the primary isobar,
- ℓ' is the orbital angular momentum, between the bachelor particle (π^-) and the isobar $[a_0(980)]$ for the secondary decay,
- σ is the spin of the secondary isobar,
- J is the spin of the $K^+\bar{K}^0\pi^-\pi^-$ system,
- P is the parity of the $K^+\bar{K}^0\pi^-\pi^-$ system, given by $P = (-1)^{\ell+s}$,
- G is the G-parity of the $K^+\bar{K}^0\pi^-\pi^-$ system given by $G = C(-1)^I$ where I is the isospin of the $K^+\bar{K}^0\pi^-\pi^-$ system,
- m is the absolute value of the projection of J on the quantization axis [the z-axis in the primary decay (Gottfried-Jackson) frame], and
- w and w' denote the primary and secondary isobar masses.

[1] Although the $f_1(1285)$ is below the threshold of $K^*(892)K$ production, the decay mode $f_1(1285) \to K^*(892)K$ was also tried and turned out to be negligible.

Since parity is conserved in strong interactions, the parity conservation in the production process was taken into account through change of basis via a reflection operator ϵ defined in the overall CM system (see Section 3).

Partial Wave Amplitudes

The intensity distribution of the reaction $\pi^- p \rightarrow K^+ \bar{K}^0 \pi^- \pi^- p$ is given by

$$I(\tau) = \sum_{bb'\epsilon} \rho^{\epsilon}_{bb'} A^{\epsilon}_{b} A^{\epsilon}_{b'}{}^{*} \tag{1.3}$$

where b is a set of quantum numbers given by Eq. (1.2), and A^{ϵ}_{b} is a "decay amplitude". $\rho^{\epsilon}_{bb'}$ is the spin-density matrix given by

$${}^{\epsilon}\rho_{bb'} = \sum_{k}^{K} {}^{\epsilon}V_{bk} \; {}^{\epsilon}V^{*}_{b'k} \tag{1.4}$$

where V_{bk} are complex parameters ("production amplitude") which we want to find from the data by using the decay amplitudes in the fit, and k allows for spin flip/spin non-flip of the proton. The parametrization of the spin density matrix was that of Chung and Trueman [6] which ensured that the conditions of positivity, parity conservation, and rank two were automatically satisfied with the minimum number of parameters.

The decay amplitude A is the function of the eight variables, which are six angles $[\Omega = (\Theta, \Phi)$, $\Omega_h = (\theta, \phi)$, and $\Omega'_h = (\theta', \phi')]$ and the primary and secondary isobar masses (w, w'). The angular dependent factors of the decay amplitudes are described as a complete set of orthonormal functions spanning the six angles. Two Breit-Wigner functions and three Blatt-Weisskopf centrifugal-barrier factors are used to describe the mass/momentum dependent factors of the decay ampli-

Table 1.1: Allowed waves for the $f_1(1285)\pi^-$and $\eta(1295)\pi^-$systems.

$I^G(J^{PC}) \setminus$ decay	$f_1(1285)\pi$	$\eta(1295)\pi$
$1^-(0^{++})$	P-wave	S-wave
$1^-(1^{++})$	P-wave	
$1^-(1^{-+})$	S, D-wave	P-wave
$1^-(2^{++})$	P, F-wave	D-wave
$1^-(2^{-+})$	D-wave	
$1^-(3^{++})$	F-wave	
$1^-(3^{-+})$	D, G-wave	F-wave
$\vdots$	$\vdots$	$\vdots$

tudes. The explicit form of decay amplitude is discussed in Section 4. The purpose of the PWA is to parameterize the production amplitude as a function of those eight variables (six angles and two isobar masses) by fitting a data set for each bin with given mass and t.

Allowed Waves

The waves in table 1.1 are allowed by the spin-parity conservation for the $f_1(1285)\pi^-$and $\eta(1295)\pi^-$systems. Here, J^{PC}=1^{-+} and 3^{-+} are exotic quantum numbers. An incoherent phase-space-like, isotropic background is also included. Since the waves are orthogonal, contributions from waves which have not been included are attributed to the background.

Reggeons

The reaction $\pi^- p \rightarrow K^+ \bar{K}^0 \pi^- \pi^- p$ is a neutral exchange process. If

Table 1.2: Reggeons expected to be exchanged for the reaction.

$\epsilon \setminus J^{PC}$	0^{++}	1^{++}	1^{-+}	2^{++}	2^{-+}	...
$+$		f_2, ρ, f_0	f_2, ρ	f_2, ρ	f_2, ρ, f_0	...
$-$	η	b_1, f_1	b_1, f_1, η	b_1, f_1, η	b_1, f_1	...

several Regge trajectories (called "Reggeons") can be exchanged in the process, then the highest trajectory will dominate at any given t [7]. Not considering a diffractive reaction through a pomeron exchange, the Reggeons expected to be exchanged for the reaction are shown in table 1.2 (in the order of the height of the trajectories). Here ϵ is a "reflectivity" which is defined to coincide with the naturality of exchanged Regge trajectories as

$$\epsilon = P_{ex}(-1)^{J_{ex}}$$

where $J_{ex}(P_{ex})$ is the spin (parity) of the particle exchanged by the beam and target particles to produce the two body-final state. The "natural" parity sequence is formed by multiplets with $P = (-1)^J$, i.e., $0^+, 1^-, 2^+, \ldots$; the "unnatural" sequence has $P = (-1)^{J+1}$. π exchange is not allowed by the G-parity conservation.

Proton Spin

The relative spin of the target proton and the recoil proton was unknown since this experiment did not measure the polarization of either. There were, however, two possibilities: "spin flip" and "spin non-flip". One wave was chosen to define an axis in spin flip space for each reflectivity. The result of this choice was that whatever combination of spin flip and spin non-flip produces the defining wave, all other waves were measured against it. The statistics of the data sample were rather small to include both spin flip and spin non-flip, the assumption that the spin of recoil proton was the same as the spin of the target proton was made, i.e., only $k = 1$ was used. This assumption reduced the number of fitting parameters by a factor of a half. As a check, some waves were allowed for $k = 2$, and in those cases, and the $k = 2$ parameters were negligibly small.

Phase Shift

For a simple Breit-Wigner resonance without background, we expect to see a strong shift (~ 180 degree) in the phase α_1 of the resonance state as we go through the peak of the enhancement. In any real case, the variation of the phase α_2 of the non-resonance reference state can distort the behavior of $\alpha_2 - \alpha_1$ even for a simple Breit-Wigner resonance. If the phase is nearly constant, it implies that both wave (the reference wave and the wave to be measured) are resonating or that neither is resonating. If the phase increases by 180 degree, it implies that reference wave is resonating and if the phase decreases by 180 degree, it implies that the wave measured is resonating. The effect of phase motion by more than 180 degree can be explained by the presence of more than one overlapping resonances.

2 PWA Program

Overview of the PWA program

The PWA program divided naturally into two parts. The first part calculated decay amplitudes and normalization integrals of these decay amplitudes over relevant regions of phase-space and prepared all the necessary quantities needed by the second part of the program. The second part of the program was a fitting program to parametrize density matrix elements which we wanted to find from the data by using the decay amplitudes and normalization integrals. The parameterization of the density matrix elements was done by an "extended maximum likelihood method".

The decay amplitude calculated from the data events then formed part of the input into the likelihood fitting program. The Monte-Carlo events were used to compute the other inputs into fit, namely the "normalization integrals" and "accepted normalization integrals" (see Section 8). The acceptance, however, was not known analytically and determined by a Monte-Carlo procedure in which events were generated and the effects of the spectrometer were then modeled to determine the acceptance as a function of the kinematical variables describing the event. The integration was done over each mass bin chosen for the analysis for the accepted Monte-Carlo events and the generated Monte-Carlo events. For the likelihood fits, only the accepted normalization integrals were necessary, however the (generated) normalization integrals were used after the fit to determine the acceptance corrected wave intensities.

The likelihood function for finding n events of a given bin with a finite acceptance $\eta(\tau)$ was defined as a product of the probabilities,

$$\mathcal{L} \propto \; [\frac{\bar{n}^n}{n!}\, \mathrm{e}^{-\bar{n}}] \; \prod_i^n [\frac{I(\tau_i)}{\int I(\tau)\eta(\tau)\; pq\; d\tau}]$$

where the first bracket is the Poisson probability for n events, and

$I(\tau)$ is the full intensity distribution given by Eq. (1.3).

The relevant parameters for PWA are collectively denoted by τ,

$$\begin{aligned}
\tau &= (\Omega, \Omega_h, \Omega'_h, w, w') \\
d\tau &= d\Omega\ d\Omega_h\ d\Omega'_h\ dw\ dw' \\
&= d\Phi\ d\cos\Theta\ d\phi\ d\cos\theta\ dw\ d\phi'\ d\cos\theta'\ dw'.
\end{aligned}$$

The principle of extended maximum likelihood maintains that the best estimate of the production amplitude ${}^{\epsilon}V_{bk}$ in Eq. (1.4) is that which maximize likelihood $\mathcal{L}$.

3 Definitions of Reference Frames and Angles

A full description of the four-body system requires eight variables; we have chosen the six angles $\Omega = (\Theta, \Phi)$, $\Omega_h = (\theta, \phi)$ and $\Omega'_h = (\theta', \phi')$ and the isobar masses w and w' as these variables. We choose the angles $\Omega = (\Theta, \Phi)$ to be the polar and azimuthal angles describing the isobar orientation $\vec{p}$ in the Gottfried-Jackson (G-J) frame (X^- rest frame), i.e., the z-axis is chosen to be parallel to the beam momentum in the X^- rest frame and the y-axis is along the production normal $\vec{beam} \times \vec{X}^-$, defined in the overall center-of-mass system. The angles $\Omega_h = (\theta, \phi)$ specify the orientation $\vec{q}$ of one of the decay products of the isobar in the $K\bar{K}\pi$ helicity frame (isobar rest frame), in which the axes are defined such that $\hat{z}_h = \hat{p}$ and $\hat{y}_h \propto\ \hat{z} \times\ \hat{p}$ and $\hat{z}$ is the z-axis in the G-J frame ("$\hat{\ }$" signifies a unit vector). The angle $\Omega'_h = (\theta', \phi')$ is defined in the same manner, i.e. the angles $\Omega'_h = (\theta', \phi')$ specify the orientation $\vec{q}'$ of one of the decay products of the isobar in the second helicity frame [$K\bar{K}$ or $a_0(980)$ isobar rest frame], in which the axes are defined such that $\hat{z}'_h = \hat{q}$ and $\hat{y}'_h \propto \hat{z}_h \times \hat{q}$ and $\hat{z}_h$ is the z-axis in the $K\bar{K}\pi$ helicity frame.

4 Decay Amplitudes

The description of particle decays into two other particles is most compactly given within the helicity formalism in which the rotation functions $D^J_{mm'}$ are used for the angular dependence. Then the amplitude for the X^- decay into the isobar, followed in turn by its decays, is given by [8]

$$A_b(\tau) \propto \tilde{J}\tilde{s}\tilde{\sigma} \sum_{\lambda,\nu} D^{J\,*}_{m\lambda}(\Phi,\Theta,0)\; D^{s\,*}_{\lambda\nu}(\phi,\theta,0)\; D^{\sigma\,*}_{\nu 0}(\phi',\theta',0)\; f^a_{\lambda\nu}(w,w') \tag{4.5}$$

where

$$\tilde{J} = \sqrt{(2J+1)}, \qquad \tilde{s} = \sqrt{(2s+1)}, \qquad \tilde{\sigma} = \sqrt{(2\sigma+1)}$$

s and λ are the spin and helicity of the isobar for the primary decay, σ and ν are the spin and helicity of the isobar for the secondary decay, and $f^a_{\lambda\nu}(w,w')$ is the helicity coupling constant (but it depends on the isobar masses w and w' of the primary isobar v and the secondary isobar v', respectively). a and b are defined by Eq. 1.1 and 1.2. It is related to the ℓs-coupling constants g_a through a standard prescription [8],

$$f^a_{\lambda\nu}(w,w') \propto \frac{\tilde{\ell}\tilde{\ell}'}{\tilde{J}\tilde{s}}(\ell 0 s\lambda|J\lambda)(\ell' 0\sigma\nu|s\nu)\; Q^v_{\ell s}(w)\; Q^{v'}_{\ell'\sigma}(w')\; g_a \tag{4.6}$$

In the isobar model, g_a is considered independent of w, w', but it can depend on the W. In partial wave analyses, g_a is in fact absorbed into the production amplitudes, which are then the parameters to be determined in maximum likelihood fits.

The factor Q contains all the "known" dependence on the isobar masses w, w':

$$
\begin{aligned}
Q^{v}_{\ell s}(w) &= F_{\ell}(p)\Delta_v(w) \\
Q^{v'}_{\ell'\sigma}(w') &= F_{\ell'}(q)F_{\sigma}(q')\Delta_{v'}(w')
\end{aligned}
$$

where the functions F denote the angular-momentum barrier factors and are defined below. The Δ-functions have the standard Breit-Wigner forms for the isobar v and v',

$$
\begin{aligned}
\Delta_v(w) &= \frac{w_0\Gamma_0}{w_0^2 - w^2 - i w_0 \Gamma_v(w)}, \\
\Delta_{v'}(w') &= \frac{w_0'\Gamma_0'}{w_0'^2 - w'^2 - i w_0' \Gamma_{v'}(w')}
\end{aligned}
$$

with

$$
\Gamma_v(w) = \Gamma_0 \frac{w_0}{w}\frac{q}{q_0}\frac{F^2_{\ell'}(q)}{F^2_{\ell'}(q_0)}, \quad \Gamma_{v'}(w') = \Gamma_0' \frac{w_0'}{w'}\frac{q'}{q_0'}\frac{F^2_{\sigma}(q')}{F^2_{\sigma}(q_0')}
$$

where w_0 and Γ_0 are the mass and the width of the isobar and $q_0 = q(w_0)$ and w_0' and Γ_0' are the mass and the width of the isobar, and $q_0' = q'(w_0)$ so that $\Gamma_v(w_0) = \Gamma_0$ and $\Delta_v(w_0) = 1$ and $\Gamma'_{v'}(w_0) = \Gamma_0'$ and $\Delta'_{v'}(w_0') = 1$. Note that the w and w' dependences of the widths of Breit-Wigner functions are assumed to be given by the two-body phase-space factors q/w and q'/w' with the functions given by $F_{\ell'}(q)$ and $F_{\sigma}(q')$, respectively.

The functions $F_\ell(p)$, $F_{\ell'}(q)$ and $F_\sigma(q')$ are the Blatt-Weisskopf centrifugal-barrier factors as given by von Hippel and Quigg [10],

$$
\begin{aligned}
F_0(p) &= 1 \\
F_1(p) &= \sqrt{\frac{2z}{z+1}} \\
F_2(p) &= \sqrt{\frac{13z^2}{(z-3)^2+9z}} \\
F_3(p) &= \sqrt{\frac{277z^3}{z(z-15)^2+9(2z-5)^2}} \\
F_4(p) &= \sqrt{\frac{12746z^4}{(z^2-45z+105)^2+25z(2z-21)^2}}
\end{aligned}
$$

where $z = (p/p_R)^2$ and $p_R = 0.1973\,(\mathrm{GeV/c})$ corresponding to 1 fermi. Normally, the barrier factor is defined such that $F_\ell(p) \Rightarrow\ 1$ as $p \Rightarrow\ \infty$, but it here is renormalized so that $F_\ell(p) = 1$ for $z = 1$, in order to have the fitted parameter for the ℓ wave reflect its "actual" size at $p = p_R$. Note that

$$
\begin{aligned}
A_b &\propto\ p^\ell \qquad \text{for } p \approx\ 0, \\
A_b &\propto\ q^{\ell'} \qquad \text{for } q \approx\ 0,
\end{aligned}
$$

and

$$A_b \propto\ q'^\sigma \qquad \text{for } q' \approx\ 0.$$

In other words, we demand that the amplitude A_b has the "correct" p, q and q' dependence near threshold.

Combining Eq. (4.5) and Eq. (4.6), one may separate out explicitly the w dependence from that of the angles and write, dropping the constant g_a,

$$A_b(\tau) = E_m^{J\ell s\ell'\sigma\ *}(\Omega, \Omega_h, \Omega'_h)\ Q^v_{\ell s}(w)\ Q^{v'}_{\ell'\sigma}(w')$$

where the functions E, which has no explicit dependence on the isobars v and v', form a complete orthonormal set in the space spanned by the six angles Ω , Ω_h and Ω'_h:

$$\begin{aligned} E_m^{J\ell s\ell'\sigma\ *}(\Omega,\Omega_h,\Omega'_h) &= \tilde{J}\tilde{s}\tilde{\sigma}\sum_{\lambda,\nu} D^{J\ *}_{m\lambda}(\Phi,\Theta,0)\ D^{s\ *}_{\lambda\nu}(\phi,\theta,0)\ D^{\sigma\ *}_{\nu 0}(\phi',\theta',0) \\ &\times \quad [\frac{\tilde{\ell}\tilde{\ell}'}{\tilde{J}\tilde{s}}(\ell 0 s\lambda|J\lambda)(\ell' 0\sigma\nu|s\nu)] \end{aligned}$$

Note that

$$E_m^{J\ell s\ell'\sigma\ *}(\Omega,\Omega_h,\Omega'_h) = 1, \qquad \text{if } J=\ell=\ell'=s=\sigma=m=0$$

and also that

$$\sum_\lambda [\frac{\tilde{\ell}}{\tilde{J}}(\ell 0 s\lambda|J\lambda)]^2 = 1$$

and

$$\sum_\nu [\frac{\tilde{\ell}'}{\tilde{s}}(\ell' 0\sigma\nu|s\nu)]^2 = 1.$$

It can be shown that

$$\begin{aligned} &\int d\Omega\ d\Omega_h\Omega'_h\ E_{m_1}^{J_1\ell_1 s_1\ell'_1\sigma_1\ *}(\Omega,\Omega_h,\Omega'_h)\ E_{m_2}^{J_2\ell_2 s_2\ell'_2\sigma_2\ *}(\Omega,\Omega_h,\Omega'_h) \\ = &\quad (4\pi)^3\ \delta_{J_1J_2}\ \delta_{\ell_1\ell_2}\ \delta_{s_1s_2}\ \delta_{\ell'_1\ell'_2}\ \delta_{\sigma_1\sigma_2}\ \delta_{m_1m_2}. \end{aligned}$$

Formally, the functions E are related to the "ket" states $|\ \ell s\ell'\sigma Jm\rangle$ and $|\ \Omega,\Omega_h,\Omega'_h\rangle$ via

$$\langle\Omega,\Omega_h,\Omega'_h\ |\ \ell s\ell'\sigma Jm\rangle = E_m^{J\ell s\ell'\sigma\ *}(\Omega,\Omega_h,\Omega'_h)$$

with the "ket" states normalized according to

$$\langle\ell'_1\sigma_1\ell_1 s_1 J_1 m_1\ |\ \ell'_2\sigma_2\ell_2 s_2 J_2 m_2\rangle = \delta_{\ell_1\ell_2}\ \delta_{s_1s_2}\ \delta_{\ell'_1\ell'_2}\ \delta_{\sigma_1\sigma_2}\ \delta_{J_1J_2}\ \delta_{m_1m_2}$$

and

$$\langle \Omega_1, \Omega_{h1}, \Omega'_{h1} \mid \Omega_2, \Omega_{h2}, \Omega'_{h2} \rangle = (4\pi)^3\, \delta(\Omega_1 - \Omega_2)\, \delta(\Omega_{h1} - \Omega_{h2}) \delta(\Omega'_{h1} - \Omega'_{h2}).$$

It is seen that the E functions have "rational" normalizations in the sense that the "ket" states $\mid \Omega\rangle$, $\mid \Omega_h\rangle$ and $\mid \Omega'_h\rangle$ are each normalized to its full 4π steradians.

5 Reflectivity Basis

The parity conservation in the production process is taken into account through change of basis via a reflection operator [6] Π_y defined in the $K^+\bar{K}^0\pi^-\pi^-$ production plane (overall CM system) as

$$\Pi_y|JM\epsilon\rangle = -\eta|JM\epsilon\rangle.$$

with the new basis involves introduction of the reflectivity ϵ through

$$\mid \epsilon am\rangle = [\mid am\rangle - \epsilon P(-1)^{J-m} \mid a - m\rangle]\theta(m)$$

where P is the parity of the state a and

$$\begin{aligned} \theta(m) &= \frac{1}{\sqrt{2}}, \qquad m > 0 \\ &= \frac{1}{2}, \qquad m = 0 \\ &= 0, \qquad m < 0. \end{aligned}$$

The new representation in this basis have two advantages [6]. First, the density matrix ${}^{\epsilon}\rho$ breaks up into submatrices ${}^{(+)}\rho$ and ${}^{(-)}\rho$ in block diagonal form; i.e.,

$${}^{\epsilon}\rho = \begin{pmatrix} {}^{(+)}\rho & 0 \\ 0 & {}^{(-)}\rho \end{pmatrix}.$$

Constraints due to parity conservation are thereby explicitly and completely taken into account. Secondly, the reflectivity ϵ is here defined such that it coincides with the naturality of exchanged Regge trajectories. For $m = 0$, $\epsilon = -1$ corresponds to a $K^+\bar{K}^0\pi^-\pi^-$ state being formed by natural parity ($J^P = 0^+, 1^-, 2^+, \ldots$) exchange while $\epsilon = 1$ corresponds to formation by un-natural parity ($J^P = 0^-, 1^+, 2^-, \ldots$) exchange. In the new basis, the E functions can be written as

$$\begin{aligned} {}^{\epsilon}E_m^{J\ell s\ell'\sigma\ *}(\Omega, \Omega_h, \Omega'_h) &= \tilde{\ell}\tilde{\ell}'\tilde{\sigma} \sum_{\lambda,\nu} D_{m\lambda}^{JP*}(\Phi, \Theta, 0)\ D_{\lambda\nu}^{s*}(\phi, \theta, 0)\ D_{\nu 0}^{\sigma\ *}(\phi', \theta', 0) \\ &\times \quad (\ell 0 s\lambda|J\lambda)(\ell' 0\sigma\nu|s\nu) \end{aligned}$$

where

$${}^{\epsilon}D_{m\lambda}^{JP\ *}(\Phi, \Theta, \phi) = \theta(m)[D_{m\lambda}^{J\ *}(\Phi, \Theta, 0) - \epsilon P(-1)^{J-m} D_{-m\lambda}^{J\ *}(\Phi, \Theta, 0)]$$

where $P = (-1)^{\ell+s}$. The normalization for the functions E in the new basis is given by

$$\begin{aligned} &\int d\Omega\ d\Omega_h\ d\Omega'_h\ {}^{\epsilon_1}E_{m_1}^{N1\ *}(\Omega, \Omega_h, \Omega'_h)^{\epsilon_2}E_{m_2}^{N2\ *}(\Omega, \Omega_h, \Omega'_h) \\ = \quad &(4\pi)^3 \delta_{\epsilon_1\epsilon_2}\delta_{J_1J_2}\ \delta_{\ell_1\ell_2}\ \delta_{s_1s_2}\delta_{\ell_1\ell_2}\delta_{\sigma_1\sigma_2}\delta_{m_1m_2} \end{aligned}$$

where $N1 = J_1\ell_1 s_1\ell'_1\sigma_1$ and $N2 = J_2\ell_2 s_2\ell'_2\sigma_2$.

Thus the final decay amplitude in the new basis is then written as

$${}^{\epsilon}A_b(\tau) = {}^{\epsilon}E_m^{J\ell s\ell'\sigma\ *}(\Omega, \Omega_h, \Omega'_h)\ Q_{\ell s}^{v}(w)\ Q_{\ell'\sigma}^{v'}(w').$$

It is in fact in this form that the decay amplitudes have been coded into the PWA program.

6 Production Amplitudes

The differential cross section for the reaction $\pi^- p \to K^+ \bar{K}^0 \pi^- \pi^- p$ can now be written, suppressing the factors which depend only on $\sqrt{s}$ and the W,

$$\frac{d\sigma}{dt\ dW\ d\tau} \propto |\mathcal{M}|^2 pqq'$$

where the variables relevant for partial wave analyses are collectively denoted by τ,

$$\begin{aligned} \tau &= \{\Omega, \Omega_h, w, \Omega_h', w'\} \\ d\tau &= d\Omega\ d\Omega_h\ d\Omega_h'\ dw\ dw' \\ &= d\Phi\ d\cos\Theta\ d\phi\ d\cos\theta\ dw\ d\phi'\ d\cos\theta'\ dw'. \end{aligned}$$

The purpose of the partial wave analysis is to parameterize the invariant amplitude $\mathcal{M}$ as a function of τ in terms of a superposition of partial waves and analyze the data for each bin with given t and W to determine the parameters which give the best description ("fit") of the data.

The intensity distribution, $I(\tau)$, is written as

$$I(\tau) = |\mathcal{M}|^2 = \sum_{\epsilon b b'} {}^{\epsilon}\rho_{bb'}\ {}^{\epsilon}A_b(\tau)\ {}^{\epsilon}A^*_{b'}(\tau).$$

The parameters V of the fit are those that enter into the definition of the spin-density matrix ρ:

$${}^{\epsilon}\rho_{bb'} = \sum_{k}^{K} {}^{\epsilon}V_{bk}\ {}^{\epsilon}V^*_{b'k}$$

where the K is the rank of the density matrix, referring to spin non-flip and spin flip of the proton ($K = 1$ in this analysis). The density matrix becomes block-diagonal, owing to parity conservation in the production process, in the sense that ϵ with a prime does not appear its definition. The V, called the production amplitude, is given in the triangular representation [6],

$$\begin{aligned} {}^{\epsilon}V_{bk} &= 0, \qquad k > n_{\epsilon}(b) \\ &= \text{real}, \qquad k = n_{\epsilon}(b) \\ &= \text{complex}, \qquad \text{otherwise} \end{aligned}$$

where $n_{\epsilon}(b)$ is an ordinal number ranging from 1 to $N_{\epsilon}(b)$ which is the total number of different states b for a given ϵ.

7 Parametrization for the Isobars

A P-wave Breit-Wigner function was used to describe the $f_1(1285) \rightarrow a_0\pi^-$ and S-wave Breit-Wigner function was used to describe the $\eta(1295) \rightarrow a_0\pi^-$. The $f_1(1285)$ and $\eta(1295)$ mass dependent factors are parametrized using nominal values [11]. The $a_0(980)$ is a resonance below the $K\bar{K}$ threshold and the decay amplitude $a_0(980) \rightarrow K\bar{K}$ is poorly known. The mass and width of the $a_0(980)$ are known from the $\eta\pi$ channel. We used the coupled-channel model [12] with one modification: the m dependence of the width is assumed to be given by the Lorentz-invariant phase-space factor q'/m, instead of merely q':

Then, the Breit-Wigner formula for the $a_0(980)$ isobar can be written

$$\Delta_{\delta}(w) = \frac{|D_{\delta}(w_t)|}{D_{\delta}(w)} \tag{7.7}$$

where

$$D_\delta(w) = w_\delta^2 - w^2 - iw_\delta \frac{w_t}{w}(g_1^2 q_1 + g_2^2 q_2).$$

The parameters g_1 and g_2 are the dimensionless coupling constants and q_1 and q_2 are the breakup momenta for the $\pi\eta$ and $K\bar{K}$ channels, respectively. The constant w_t is the $K\bar{K}$ threshold, so that the magnitude of the Δ-function is normalized to one at the threshold.

Let w_0 and Γ_0 be the mass and the width as observed in the $\pi\eta$ channel. Then, by demanding that Eq. (7.7) approximates these parameters

$$\Gamma_\delta(w) \rightarrow \;\; w^2 - w_0^2 - iw_0\Gamma_0, \qquad \text{as} \quad w \rightarrow \;\; w_0$$

with $q_2 = ik_2$, where k_2 is the magnitude of q_2 below the $K\bar{K}$ threshold, one obtains

$$w_\delta^2 = w_0^2 - r\frac{k_2(w_0)}{q_1(w_0)} w_0 \Gamma_0$$

and

$$g_1^2 = \frac{w_0^2 \Gamma_0}{w_t w_\delta q_1(w_0)}.$$

Here one has set $r = g_2^2/g_1^2$.

The shape of the resonance peak as given by Eq. (7.7) is highly asymmetric near w_t; so in practice one has to increase Γ_0 or decrease r or both to approximate a simple Breit-Wigner form of a given mass w_0 and width Γ_0. The parameters used in this analysis were $w_0 = 978$ MeV/c^2, $r = 0.5$, and $\Gamma_0 = 68$ MeV/c^2.

8 Maximum-likelihood Methods

The PWA of the $K^+\bar{K}^0\pi^-\pi^-$ system is done by using the extended maximum likelihood method [13, 14] to determine the parameters for the production amplitudes V.

The probability for the i-th event in the data sample can be given by

$$P_{obs}^{(i)} \;=\; P^{(i)}\;\mathcal{A}^{(i)} \tag{8.8}$$

where $P^{(i)}$ is the theoretical probability ("physics") for the i-th event to occur, and $\mathcal{A}^{(i)}$ is the probability that it is observed in the detector and passes the selection criteria ("acceptance").

Consider a sample of n observed events in a bin of area Δm. The likelihood function may be written as a product of a possibilities:

$$\mathcal{L} = \prod^{n} P_{obs}^{(i)} \tag{8.9}$$

In the usual likelihood approach we determine parameters relevant to the shapes of distributions, but simply take the absolute normalizations as being equal to the observed number of events. But the extended maximum likelihood method also provides estimates of the absolute normalization. The "extended" likelihood function $\mathcal{L}$ includes an extra factor for the probability of obtaining a sample of size n from a Poisson distribution of $\bar{n}$. [13]

Then, the likelihood function for finding n events of a given bin with a finite acceptance $\eta(\tau)$ can be written

$$\mathcal{L} \propto \;[\frac{\bar{n}^n}{n!}\;\mathrm{e}^{-\bar{n}}]\;\prod_{i}^{n}\,[\frac{I(\tau_i)}{\int I(\tau)\eta(\tau)\;pqq'\;d\tau}] \tag{8.10}$$

where the first bracket is the Poisson probability for n events. This is the so-called extended likelihood function, in the sense that the Poisson distribution for n itself is included in the likelihood function, i.e., it is one of the parameters to be determined.

The expectation value $\bar{n}$ for n is given by

$$\bar{n} \propto \int I(\tau) \frac{\eta(\tau)}{\eta_x} \; pqq' \; d\tau$$

where η_x is the MC acceptance for the bin (defined below), so that $\eta(\tau)/\eta_x$ becomes a "renormalized acceptance function". The likelihood function $\mathcal{L}$ can now be written, dropping the factors depending on n alone,

$$\mathcal{L} \propto \; [\prod_i^n \; I(\tau_i)] \exp[-\int I(\tau) \frac{\eta(\tau)}{\eta_x} \; pqq' \; d\tau].$$

The log of the likelihood function now has the form,

$$\ln \mathcal{L} \propto \; \sum_i^n \; \ln I(\tau_i) \; - \int I(\tau) \frac{\eta(\tau)}{\eta_x} \; pqq' \; d\tau.$$

We shall adopt the following shorthand notation for indices:

$$\begin{aligned} \alpha &= \{\epsilon k c\} = \{\epsilon k a \mu m\} \\ \alpha' &= \{\epsilon k c'\} = \{\epsilon k a' \mu' m'\} \end{aligned}$$

where μ stands for either G- or C-parity. Note that primes do not appear for ϵ and k; they apply to non-interfering indices.

Now the distribution functions $I(\tau)$ assume the following compact form,

$$I(\tau) = \sum_{\alpha\alpha'} \; V_\alpha \; V^*_{\alpha'} \; \psi_\alpha(\tau) \; \psi^*_{\alpha'}(\tau)$$

where ψ is the symmetrized decay amplitude A as defined previously. As such, ψ does not depend on the index k.

The normalization integrals are most expeditiously obtained through the Monte-Carlo (MC) events. Let M be the number of MC events generated, and let M_x be the number accepted by the finite geometry of the experiment and other software cuts. The MC acceptance introduced before is given by $\eta_x = M_x/M$. One needs two sets of normalizations,

$$\Psi_{\alpha\alpha'} = \frac{1}{M}\sum_{i}^{M} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \tag{8.11}$$

for the full MC sample and

$$\Psi^x_{\alpha\alpha'} = \frac{1}{M_x}\sum_{i}^{M_x} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i)$$

for the accepted MC sample, so that

$$\eta_x \Psi^x_{\alpha\alpha'} = \frac{1}{M}\sum_{i}^{M_x} \psi_\alpha(\tau_i)\ \psi^*_{\alpha'}(\tau_i) \tag{8.12}$$

is the true accepted normalization integral, obtained by replacing M_x in the denominator with M in Eq. (8).

The log of the extended likelihood function $\mathcal{L}$ can therefore be written

$$\ln\mathcal{L} = \sum_{i}^{n} \ln[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \psi_\alpha(\tau_i)\psi^*_{\alpha'}(\tau_i)] - \eta_x[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi^x_{\alpha\alpha'}] \tag{8.13}$$

where n is the number of experimental events in a given bin. With an extended likelihood function, the parameters themselves now have an absolute normalization. It can be shown that, by substituting Vs by cV's where c is a constant independent of α, and then by differentiating

$\ln \mathcal{L}$ by c^2, that the Vs are normalized according to

$$\eta_x \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \ \Psi^x_{\alpha\alpha'} = n. \tag{8.14}$$

(The partial derivative should equal to zero at the minimum.)
The predicted number of events from the fit is

$$N = \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \ \Psi_{\alpha\alpha'} \tag{8.15}$$

One may renormalize the parameter via

$$V \to \sqrt{\frac{n}{\eta_x}}\, V$$

Then, in terms of the new parameters, the log of the likelihood function reads

$$\ln \mathcal{L} = \sum_i^n \ln[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \ \psi_\alpha(\tau_i)\psi^*_{\alpha'}(\tau_i)] - n[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \ \Psi^x_{\alpha\alpha'}] \tag{8.16}$$

and the normalization condition assumes the form

$$\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \ \Psi^x_{\alpha\alpha'} = 1. \tag{8.17}$$

With this normalization, the fitted Vs are, to first order, independent of variations among different bins, and a set of fitted parameters in a given bin is, therefore, an excellent starting value for the neighboring bins. The predicted number of events is now given by, in terms of the new parameters,

$$N = \frac{n}{\eta_x} \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \ \Psi_{\alpha\alpha'} \tag{8.18}$$

Note that $\eta_x = M_x/M$ would be the predicted fraction of accepted events if the data were distributed isotropically as in MC events. This formula shows that the "acceptance-corrected" event number N is proportional to the observed event n divided by η_x— the summation involving the fitted parameters Vs is precisely what is needed to "correct" for the anisotropy inherent in the experimental data.

9 Combining Two Triggers

In the experiment E818, we took two separate triggers, the 1-3-5 and 1-5-5 triggers, result in the same reaction [1]. The analysis was setup to fit the 1-3-5 and 1-5-5 events simultaneously, taking into account the differences in the acceptance and keeping the parameters independent of the numbers of the events to first order. The joint likelihood of the 1-3-5 and 1-5-5 triggers is simply the product of the two likelihoods $\mathcal{L}_{35}$ and $\mathcal{L}_{55}$. Then the log of the likelihood is given by

$$\ln \mathcal{L} = \sum_{i}^{n_{35}+n_{55}} \ln[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \psi_\alpha(\tau_i)\psi^*_{\alpha'}(\tau_i)] - \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, [\eta_1 \Psi^{(35)}_{\alpha\alpha'} + \eta_2 \Psi^{(55)}_{\alpha\alpha'}] \tag{9.19}$$

where the subscripts and superscripts 35 and 55 stand for the 1-3-5 and 1-5-5 triggers, respectively. The normalizations are found to be, using the techniques as before,

$$\eta_{35} \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi^{(35)}_{\alpha\alpha'} = n_{35} \tag{9.20}$$

$$\eta_{55} \sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'} \, \Psi^{(55)}_{\alpha\alpha'} = n_{55} \tag{9.21}$$

and the predicted number of events is again given by Eq. (8.15).

The parameters can be redefined in a manner similar to Eq. (8), as follows:

$$V \rightarrow \sqrt{\frac{n_{35} + n_{55}}{\eta_{35} + \eta_{55}}} \, V. \tag{9.22}$$

Then the new log of likelihood is given by

$$\begin{aligned}\ln\mathcal{L} &= \sum_i^{n_{35}+n_{55}} \ln[\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \psi_\alpha(\tau_i)\psi^*_{\alpha'}(\tau_i)] \\ &- (n_{35}+n_{55})\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ [\frac{\eta_{35}}{\eta_{35}+\eta_{55}}\Psi^{(35)}_{\alpha\alpha'} + \frac{\eta_{55}}{\eta_{35}+\eta_{55}}\Psi^{(55)}_{\alpha\alpha'}]\end{aligned} \tag{9.23}$$

and the new normalizations are

$$(\frac{n_{35}+n_{55}}{\eta_{35}+\eta_{55}})\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi^{(35)}_{\alpha\alpha'} = \frac{n_1}{\eta_1} \tag{9.24}$$

$$(\frac{n_{35}+n_{55}}{\eta_{35}+\eta_{55}})\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi^{(55)}_{\alpha\alpha'} = \frac{n_2}{\eta_2}. \tag{9.25}$$

The predicted number of events assume the form, in terms of the new parameters,

$$N = \frac{n_{35}+n_{55}}{\eta_{35}+\eta_{55}}\sum_{\alpha\alpha'} V_\alpha V^*_{\alpha'}\ \Psi_{\alpha\alpha'}. \tag{9.26}$$

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