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Formulae of Mass Dependence of Amplitudes in Coupled-Channel Formalism

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Introduction

We suggest to use the famous K-matrix formalism [1] for the analysis of mass dependent PWA results instead of Breit-Wigner formulae. Many authors did the mass dependent fit with only coherent sum of Breit-Wigner amplitudes.

Here we gather up the formulae which are necessary for the mass dependent partial wave analysis of two particle decay final states.

The next points are important:

1. We suggest to fit the amplitudes of $J^{PC} = 1^{-+}$ and 2^{++} states in K-matrix approach and to find a relative phase of P_+ and D_+ waves.

2. We can include some final states: $(\eta\pi)$, $(\rho\pi)$, $(\eta'\pi)$ and $(b_1/f_1\pi)$ for example. In the previous publications the authors discussed two states: $(\eta\pi)$, $(b_1/f_1\pi)$ in [2] and three states: $(\eta\pi)$, $(\rho\pi)$, $(b_1/f_1\pi)$ in [3] for only $J^{PC} = 1^{-+}$.

3. We can take into account two poles of K-matrix for each J^{PC}. It is $\pi_1(1440)$ and ${\pi'}_1(1600)$ for $J^{PC} = 1^{-+}$ and $a_2(1320)$ and ${a'}_2(1800)$ for $J^{PC} = 2^{++}$ in the region of mass up to 2.0 GeV.

4. We set the physical background as the direct two particle production. In is not necessary to be exactly the Deck-type background [3], but it must be present. The suggestion to fit the background contribution is a main point of this note.

Formalism

The main equation in P-vector approach is

$$F = (I - iK)^{-1}P \quad , \tag{1}$$

where P is a vector of the production amplitudes, K is K-matrix and F is a vector of the final state amplitude.

The Lorenz-invariant presentation of the amplitude and K-matrix is [1]

$$\begin{aligned} F &= \{\rho\}^{1/2}\hat{F}, \\ \hat{F} &= (I - i\hat{K}\rho)^{-1}\hat{P}, \\ K &= \{\rho\}^{1/2}\hat{K}\{\rho\}^{1/2}, \\ P &= \{\rho\}^{1/2}\hat{P}. \end{aligned} \tag{2}$$

The $\hat{K}$-matrix is [1]

$$\hat{K}_{ij}(m) = \sum_\alpha \left\{ \frac{\gamma_{\alpha i}\gamma_{\alpha j} m_\alpha \Gamma^0_\alpha \bar{B}^{(\alpha)}_L(q_i)\bar{B}^{(\alpha)}_L(q_j)}{m^2_\alpha - m^2} + C^{(\alpha)}_{ij}(m) \right\} \tag{3}$$

or

$$\hat{K}_{ij}(m) = \sum_\alpha m_\alpha \left\{ \frac{\sqrt{\Gamma_{\alpha i}(m)\Gamma_{\alpha j}(m)}}{(m^2_\alpha - m^2)\sqrt{\rho_i(m)\rho_j(m)}} + \frac{\sqrt{\Gamma_{bi}(m)\Gamma_{bj}(m)}}{D^{(b)}} \right\}. \tag{4}$$

Here α is a number of pole-resonance, m_α is a mass of resonance. The partial widths of resonances and the background $C^{(\alpha)}_{ij}$ are

$$\Gamma_{\alpha i}(m) = \gamma^2_{\alpha i}\Gamma^0_\alpha(\bar{B}^{(\alpha)}_L(q_i))^2\rho_i(m), \tag{5}$$

$$\Gamma_{bi}(m) = \gamma^2_{bi}\Gamma^0_\alpha(\bar{B}^{(\alpha)}_L(q_i))^2\rho_i(m), \tag{6}$$

where $\bar{B}^{(\alpha)}_L(q_i) = B_L(q_i(m))/B_L(q_i(m_\alpha))$. We see that $\bar{B}^{(\alpha)}_L(q_i) \equiv 1$ if $m = m_\alpha$.

The total width of K-matrix is equal to the sum of partial widths at $m = m_\alpha$:

$$\tilde{\Gamma}_\alpha = \sum_i \tilde{\Gamma}_{\alpha i} = \sum_i \Gamma_{\alpha i}(m_\alpha) = \Gamma^0_\alpha \sum_i \gamma^2_{\alpha i}\rho_i(m_\alpha). \tag{7}$$

The parameters $\gamma_{\alpha i}$ are the reduced widths. They are real and may be negative. If we use the normalization constrain

$$\sum_i \gamma^2_{\alpha i} = 1 \tag{8}$$

then

$$\Gamma^0_\alpha = \sum_i \frac{\tilde{\Gamma}_{\alpha i}}{\rho_i(m_\alpha)}. \tag{9}$$

In practice, it is better to avoid the normalization constrain (8) and use $\gamma_{\alpha i}$ as variables in the fit.

The parameters γ_{bi} of K-matrix background have to be fitted also.

The Blatt-Weisskopf barrier factors are

$$\begin{aligned} B_0 &= 1 \\ B_1(q) &= \sqrt{\frac{z}{1+z}} \\ B_2(q) &= \sqrt{\frac{z^2}{(z-3)^2 + 9z}}, \end{aligned} \tag{10}$$

where $z = (q/q_k)^2$, $q_k = 0.1973$ GeV.

The matrix ρ in (2) is diagonal, $\rho_{ij}(m) = \delta_{ij}\rho_i(m)$. The factor $\rho_i(m)$ takes into account the phase-space. For two particle decay it is equal to

$$\rho_i(m) = \frac{2q_i(m)}{m} = \sqrt{\left(1 - \frac{(m_1+m_2)_i^2}{m^2}\right)\left(1 - \frac{(m_1-m_2)_i^2}{m^2}\right)}, \tag{11}$$

where $q_i = q_i(m; m_1 m_2)$ is a decay breakup momentum in channel i and $\rho_i(m) \to 1$ as $m^2 \to \infty$.

The background $C_{ij}^{(\alpha)}(m)$ in K-matrix is set according to [3], but we generalize it for the nondiagonal terms

$$C_{ij}^{(\alpha)}(m) = \frac{\gamma_{bi}\gamma_{bj} m_\alpha \Gamma_\alpha^0 \bar{B}_L^{(\alpha)}(q_i) \bar{B}_L^{(\alpha)}(q_j)}{D^{(b)}}. \tag{12}$$

Here $D^{(b)}$ is a real constant. A value $1/D^{(b)}$ gives the strength of the background as a normalization factor for all channels and for the P- vector background.

The background $C_{ij}^{(\alpha)}(m)$ has a weak mass dependence of background because $q_i = q_i(m)$ has the weak mass dependence. Remind that in the work [3] there is a strong pole-type mass dependence of background.

If we want to reduce the number of free parameters we can put to zero some of $C_{ii}^{(\alpha)}$ or not use the background in the nondiagonal terms of $\hat{K}$-matrix, as it was made in [3].

The amplitude of production with the background is

$$\hat{P}_i(m) = \sum_\alpha \left\{ \frac{\gamma_{\alpha i} m_\alpha \Gamma_\alpha^0 \bar{B}_L^{(\alpha)}(q_i)}{(m_\alpha^2 - m^2)} V_\alpha + d_{bi}^{(\alpha)}(m) V_i^{(b)} \right\} \tag{13}$$

or

$$\hat{P}_i(m) = \sum_\alpha m_\alpha \left\{ \frac{\sqrt{\Gamma_\alpha^0 \Gamma_{\alpha i}(m)}}{(m_\alpha^2 - m^2)\sqrt{\rho_i(m)}} V_\alpha + \frac{\sqrt{\Gamma_\alpha^0 \Gamma_{bi}(m)}}{D^{(b)}\sqrt{\rho_i(m)}} V_i^{(b)} \right\}. \tag{14}$$

The parameters V_α and $V_i^{(b)}$ are complex. V_α governs the production of resonance α and $V_i^{(b)}$ governs the background in the production amplitudes.

The mass dependence of background $d_{bi}^{(\alpha)}(m)$ is set by [2]

$$d_{bi}^{(\alpha)}(m) = \frac{\gamma_{bi} m_\alpha \Gamma_\alpha^0 \bar{B}_L^{(\alpha)}(q_i)}{D^{(b)}}. \tag{15}$$

For calculation we can use the next final formulae:

$$\hat{K}_{ij}(m) = \sum_\alpha m_\alpha \Gamma_\alpha^0 \left\{ \frac{\gamma_{\alpha i}\gamma_{\alpha j}}{m_\alpha^2 - m^2} + \frac{\gamma_{bi}\gamma_{bj}}{D^{(b)}} \right\} \bar{B}_L^{(\alpha)}(q_i) \bar{B}_L^{(\alpha)}(q_j), \tag{16}$$

$$\hat{P}_i(m) = \sum_\alpha m_\alpha \Gamma_\alpha^0 \left\{ \frac{\gamma_{\alpha i}}{m_\alpha^2 - m^2} V_\alpha + \frac{\gamma_{bi}}{D^{(b)}} V_i^{(b)} \right\} \bar{B}_L^{(\alpha)}(q_i), \tag{17}$$

$$\hat{F}_i = \sum_j (I - i\hat{K}\rho)_{ij}^{-1} \hat{P}_j; \qquad (\hat{K}\rho)_{ln} = \hat{K}_{ln}\rho_n. \tag{18}$$

Let's remark, that the $\rho_n(m)$ in the amplitude $\hat{F}_i(m)$ may become imaginary below a given threshold.

The Lorentz-invariant amplitude of the i-th final state is [1]

$$F_i(m) = \sqrt{\rho_i(m)}\hat{F}_i(m)\theta(m - m_{th}^{(i)}) \tag{19}$$

where $m_{th}^{(i)}$ is the threshold of i-th channel.

Here we compile the last E852 results of the masses and the widths of 2^{++} and 1^{-+} resonances in table 1 from our publications and notes. We see that the width Γ_α^0 of $\pi'_1(1600)$ is very different for considered channels. It may be a consequence of large systematic errors or interference effects. So we need to use the K-matrix formalism when we do the mass dependent fit.

Table 1

Masses and widths of 2^{++} and 1^{-+} resonances

J^{PC}	Pole	Channel		m_α, MeV	Γ_α^0, MeV
2^{++}	$a_2(1320)$	$(\eta\pi^-)$	[4]	$1317 \pm 1 \pm 2$	$127 \pm 2 \pm 2$
	$a_2(1800)$	$(\eta'\pi^-)$	[6]	$1803 \pm 44 \pm 66$	$483 \pm 91 \pm 190$
	$\pi_1(1400)$	$(\eta\pi)$	[4]	$1370 \pm 16^{+50}_{-30}$	$385 \pm 40^{+65}_{-105}$
		$(\eta'\pi)$	[6]	$1378 \pm 108 \pm 24$	$380 \pm 279 \pm 148$
1^{-+}		$(\rho^0\pi)$	[5]	$1593 \pm 8^{+29}_{-47}$	$168 \pm 20^{150}_{12}$
	$\pi'_1(1600)$	$(\eta'\pi)$	[6]	$1676 \pm 40 \pm 20$	$324 \pm 69 \pm 45$
		$(b_1/f_1\pi)$	[7]	$1558 \pm 7 \pm 19$	$308 \pm 12 \pm 6$

So, we can use the next list (see tables 2 and 3) of channels and poles parameters as the input values. We don't know a real value of Γ_α^0. We'll take its input value from the pointed interval, which corresponds to the change in limit of systematical errors. The real magnitude of Γ_α^0, also as m_α, will be find from the fit.

Table 2

The poles of 2^{++} states

Channel	Final state	$m_{th}^{(i)}$	Poles:	
1	$(\eta\pi)$, $L = 2$	689 MeV		$m_1 = 1320$ MeV
			$a_2(1320)$,	$\Gamma_1^0 = 120$ MeV
2	$(\rho\pi)$, $L = 2$	910 MeV		V_1, complex and unitless
3	$(\eta'\pi)$, $L = 2$	1098 MeV		$m_2 = 1800$ MeV
			$a'_2(1800)$,	$\Gamma_2^0 = 200 \div 500$ MeV
4	$(b_1/f_1\pi)$, $L = 1$	1400 MeV		V_2, complex and unitless
5	$(\eta(1295)\pi)$, $L = 2$	1435 MeV		
6	$(\omega\rho)$, $L = 0,\ 2$	1553 MeV		

Table 3

The poles of 1^{-+} states

Channel	Final state	Poles:	
1	$(\eta\pi)$, $L=1$		$m_1 = 1400$ MeV
		$\pi_1(1400)$,	$\Gamma_1^0 = 200 \div 500$ MeV
2	$(\rho\pi)$, $L=1$		V_1, complex and unitless
3	$(\eta'\pi)$, $L=1$		$m_2 = 1600$ MeV
		$\pi'_1(1600)$,	$\Gamma_2^0 = 120 \div 450$ MeV
4	$(b_1/f_1\pi)$, $L=0,\ 2$		V_2, complex and unitless
5	$(\eta(1295)\pi)$, $L=1$		
6	$(\omega\rho)$, $L=1,\ 3$		

So we have to fit m_α and Γ_α, using the mass dependent experimental results for all considered channels.

The other fitted parameters are

$\gamma_{\alpha i}$, real, but may be negative, unitless;
γ_{bi}, real, unitless;
$D^{(b)}$, GeV^2;
V_α, $V_i^{(b)}$, complex and unitless.

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