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A generalization of the width formulas in two-body decays —Version II—

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abstract

The width formulas require modification when covariant helicity-coupling amplitudes are used to describe two-body decays.

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1 Introduction

Consider decay of a resonance with integer spin J into two daughter states of integer spins whose helicities are denoted by λ and λ'. The covariance requirement of the decay amplitudes demands that the helicity-coupling amplitudes acquire dependence on the Lorentz factors[1]. As a result, the symmetry between the helicity-coupling amplitudes $F^J_{\lambda\lambda'}$ and the ℓS-coupling amplitudes $g_{\ell S}$ is broken—where ℓ and S are the orbital angular momentum and the intrinsic total spin, respectively, for the final state in the non-relativistic limit.

This note describes how one deals with the covariance requirement in writing down the decay widths, starting at first with a concrete example of the $b_1(1235)$ decay into $\omega+\pi$. Two further examples follow.

2 $b_1(1235) \to \omega + \pi$

Let J stand for the spin of the $b_1(1235)$ and $|s\lambda\rangle$ for the final-state ω. The net intrinsic parity for this decay is given by $+1$ and $F^J_\lambda = +F^J_{-\lambda}$, and there are two allowed orbital angular momenta, i.e., $\ell = 0$ or $\ell = 2$. When the amplitudes are constructed with the aid of projection operators, the invariant helicity-coupling amplitudes are simply given by

$$\begin{aligned} \sqrt{2}F^J_+ &= \sqrt{\frac{2}{3}}g_0 + \sqrt{\frac{1}{3}}g_2 r^2 \\ F^J_0 &= \left(\sqrt{\frac{1}{3}}g_0 - \sqrt{\frac{2}{3}}g_2 r^2\right)\gamma \end{aligned} \tag{1}$$

where r is the relative momentum in the $\omega\pi$ system and γ is the Lorentz factor for the ω, given by $\gamma = E_\omega/m_\omega$ in the $\omega\pi$ rest system.

Let $\Omega(\vartheta,\varphi)$ describe the direction of the ω in the $\omega\pi$ rest system. Suppose now that ω decays into 3π and the orientation of the normal to its decay plane—in the ω rest frame—is given by $\Omega'(\vartheta',\varphi')$ in the helicity coordinate system $\{\hat{x}_h, \hat{y}_h, \hat{z}_h\}$ defined in the usual way[2]. Then the overall decay amplitude is

$$A \propto \sum_\lambda D^{J*}_{M\lambda}(\varphi,\vartheta,0) F^J_\lambda D^{s*}_{\lambda 0}(\varphi',\vartheta',0) \tag{2}$$

so that

$$\sum_M |A|^2 \propto \sum_{\substack{M \\ \lambda\lambda'}} D^{J*}_{M\lambda}(\varphi,\vartheta,0) D^J_{M\lambda'}(\varphi,\vartheta,0) F^J_\lambda F^{J*}_{\lambda'} D^{s*}_{\lambda 0}(\varphi',\vartheta',0) D^s_{\lambda' 0}(\varphi',\vartheta',0) \tag{3}$$

where $J = 1$ for the $b_1(1235)$ and $s = 1$ for the $\omega(782)$. The reader may note that the D-function describing the ω decay from a state $|s\lambda\rangle$—see (3)—has a zero for the the second index in the subscripts. A thorough discussion on this point is given in the Appendix. Integrating over the angles Ω and Ω', one finds

$$\int \mathrm{d}\Omega\,\mathrm{d}\Omega' \sum_M |A|^2 \propto \sum_\lambda |F^J_\lambda|^2 \tag{4}$$

The mass-dependent width has the following expression

$$\begin{aligned}\Gamma(w) &\propto \left(\frac{q}{w}\right) \int \mathrm{d}\Omega\,\mathrm{d}\Omega' \sum_M |A|^2 \\ &\propto \left(\frac{q}{w}\right) \sum_\lambda |F^J_\lambda|^2\end{aligned} \tag{5}$$

where w (q) stands for the effective mass (break-up momentum) of the $\omega\pi$ system. One sees that, from (1),

$$\Gamma(w) \propto \left(\frac{q}{w}\right) \left[\left(\frac{1}{3}\gamma^2 + \frac{2}{3}\right) g_0^2 + (\frac{2}{3}\gamma^2 + \frac{1}{3}) g_2^2\, r^4 \pm \frac{2\sqrt{2}}{3} g_0 g_2\, r^2 (\gamma^2 - 1)\right] \tag{6}$$

where one assumes that g_0 and g_2 are relatively real but that the relative sign could be negative. It should be emphasized that γ and r are both dependent on w. Note that one obtains the familiar result in the limit $\gamma \to 1$, i.e.

$$\Gamma(w) \propto \left(\frac{q}{w}\right) \left[g_0^2 + g_2^2\, r^4\right] \tag{7}$$

The factors r^ℓ appearing (6) and (7) arise naturally from a covariant tensor formulation of the decay amplitudes. However, one replaces them—in practice—by the Blatt-Weisskopf barrier factors[3], so that r^ℓ is damped as $w \to \infty$.

Let Γ_0 be the nominal, mass-independent width of the $b_1(1235)$ and w_0 be its mass. Denoting the square bracket in (6) by $f(r,\gamma)$, one may write

$$\Gamma(w) = \Gamma_0 \left(\frac{w_0}{w}\right) \left(\frac{q}{q_0}\right) \left[\frac{f(r,\gamma)}{f(r_0,\gamma_0)}\right] \tag{8}$$

Or, in the limit $\gamma \to 1$, one finds

$$\Gamma(w) = \Gamma_0 \left(\frac{w_0}{w}\right) \left(\frac{q}{q_0}\right) \left[\frac{g_0^2 + g_2^2\, r^4}{g_0^2 + g_2^2\, r_0^4}\right] \tag{9}$$

In both of these formulas, the subscripts 0 stand for $w = w_0$, so that one obtains $\Gamma(w_0) = \Gamma_0$. Although this procedure is not the only way to connect the mass-dependent width $\Gamma(w)$ to Γ_0, it is perhaps the simplest.

It should be emphasized that the mass-dependent width which must appear in the Breit-Wigner formula refers to the *total* width, i.e.

$$\Delta(w) = \frac{w_0\,\Gamma_0}{w_0^2 - w^2 - i\,w_0\,\Gamma(w)} \tag{10}$$

where $\Gamma(w)$ is given by (8). One sees from (6) that the formula above depends not only on g_0 and g_2 but also on their interference term.

One sees that $\gamma = 1.095$ for $b_1(1235) \to \omega + \pi$. It is therefore obvious that (7) is a reasonable approximation to (6). However, this is not the case in the next example.

3 $\pi_2(1670) \to \rho + \pi$

There are two orbital angular momenta allowed in this decay, i.e. $\ell = 1$ and $\ell = 3$. The covariant helicity-coupling amplitudes are[1]

$$\begin{aligned} \sqrt{2}F_+^J &= \sqrt{\frac{3}{5}}g_1\,r + \sqrt{\frac{2}{5}}g_3\,r^3 \\ F_0^J &= \left(\sqrt{\frac{2}{5}}g_1\,r - \sqrt{\frac{3}{5}}g_3\,r^3\right)\gamma \end{aligned} \tag{11}$$

Now the mass-dependent width is, from (5),

$$\Gamma(w) \propto \left(\frac{q}{w}\right)\,\left[\left(\frac{2}{5}\gamma^2 + \frac{3}{5}\right)g_1^2\,r^2 + (\frac{3}{5}\gamma^2 + \frac{2}{5})g_3^2\,r^6 \pm \frac{2\sqrt{6}}{5}g_1 g_3\,r^4(\gamma^2 - 1)\right] \tag{12}$$

Note that $\gamma = 1.309$ for this decay. Clearly, the Lorentz factor cannot be ignored in this case.

Upon setting $\gamma = 1$, one obtains the familiar non-relativistic result:

$$\Gamma(w) \propto \left(\frac{q}{w}\right)\,\left[g_1^2\,r^2 + g_3^2\,r^6\right] \tag{13}$$

4 $a_2(2050) \to f_2(1270) + \pi$

The initial particle is the putative ${}^3F_2$ state expected in the standard quarkonium model[4]. From parity conservation in the decay, one must have $F^J_\lambda = -F^J_{-\lambda}$, so that $F^J_0 = 0$. Again, the allowed orbital angular momenta are $\ell = 1$ and $\ell = 3$. The covariant helicity-coupling amplitudes are[1]

$$\begin{aligned} \sqrt{2}F_2^{(2)} &= -w\left(\frac{2}{\sqrt{5}}g_1 + \frac{1}{\sqrt{5}}g_3\, r^2\right)\, r \\ \sqrt{2}F_1^{(2)} &= w\gamma\left(-\frac{1}{\sqrt{5}}g_1 + \frac{2}{\sqrt{5}}g_3\, r^2\right)\, r \end{aligned} \tag{14}$$

The presence of w can be traced to the fact that a covariant description of the decay amplitude requires the totally antisymmetric four-dimensional tensor $\epsilon^{\alpha\beta\gamma\delta}$.

So the mass-dependent width becomes, from (5),

$$\Gamma(w) \propto\ (q\,w)\ \left[\left(\frac{1}{5}\gamma^2 + \frac{4}{5}\right) g_1^2\, r^2 + (\frac{4}{5}\gamma^2 + \frac{1}{5}) g_3^2\, r^6 \pm \frac{4}{5} g_1 g_3\, r^4 (\gamma^2 - 1)\right] \tag{15}$$

The non-relativistic form is obtained by setting $\gamma = 1$, i.e.

$$\Gamma(w) \propto\ (q\,w)\ \left[g_1^2\, r^2 + g_3^2\, r^6\right] \tag{16}$$

This is close to the cardinal form of (13), except that $1/w$ replaced by w. One notes that $\gamma = 1.113$. This is close enough to 1, so that one could set $\gamma = 1$ in practice—but, of course, it depends on the statistics.

Appendix

Consider the decay $\omega \to 3\pi$ from a state $|s\lambda\rangle$. Let $R(\alpha, \beta, \gamma)$ be the Euler angles describing the orientation of the 3π system in the ω rest frame. One chooses here the normal to the decay plane as the analyzer, i.e. $\alpha = \phi'$ and $\beta = \vartheta'$ in the notation of section 2. Then, the ω decay amplitude is given by

$$A_\omega \sim \sum_\mu D^{s*}_{\lambda\mu}(\varphi', \vartheta', \gamma)\, F^s_\mu \tag{17}$$

where F^s_μ is the decay helicity-coupling amplitude and μ is the 'body-fixed' spin projection along the decay normal of the ω (see Section 6 of Ref. [2]). But the parity conservation in the decay $\omega \to 3\pi$ demands that

$$F^s_\mu = (-)^\mu \, F^s_\mu \tag{18}$$

thus leaving F^s_0 as the only non-zero element. Note also that, in this case, the third element γ of the Euler angles becomes a redundant variable—which could be intergrated out, so that

$$A_\omega \sim \; D^{s*}_{\lambda 0}(\varphi', \vartheta', 0) \, F^s_0 \tag{19}$$

which is as shown in (2) except that F^s_0 was omitted (or, more precisely, it was deemed constant and absorbed into the overall constant). From the techniques developed in Ref. [1], it is clear that the helicity-coupling decay amplitude for the ω must be given by

$$F^s_0 = \epsilon^{\alpha\beta\gamma\delta} \, k^\alpha \, a^\beta \, b^\gamma \, \phi^{\delta *}(0) \tag{20}$$

where four-momenta describing $\omega \to 3\pi$ are given by $k = a + b + c$ and $\phi^\alpha(\mu)$ is the ω polarization vector. Note that (20) results from setting the decay normal along the z-axis, so that all the four-momenta in $k = a + b + c$ are now in the xy-plane. But since the decay amplitude must be Lorentz invariant, this requirement is not necessary. Let w' be the effective mass of the 3π system, then one finds

$$F^s_0 = w' |\vec{a} \times \vec{b}| \tag{21}$$

where $\vec{a}$ and $\vec{b}$ are any two 3-momenta from the three pions in the ω rest frame.

It might be appropriate at this point that one recalls the general formula for the decay width

$$w \, \mathrm{d}\Gamma = |A|^2 \, \mathrm{d}\rho(b_1 \to \omega\pi, \omega \to 3\pi) \tag{22}$$

and one defines

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}w} = \Gamma(w) \tag{23}$$

The phase-space element is given by

$$\begin{aligned} \mathrm{d}\rho &\sim \; (q\mathrm{d}w) \, \mathrm{d}\Omega \, (w'\mathrm{d}w') \, \mathrm{d}R \, \mathrm{d}E_a \mathrm{d}E_b \\ &\sim \; (q\mathrm{d}w) \, \mathrm{d}\Omega \, (w'\mathrm{d}w') \, \mathrm{d}\Omega' \, \mathrm{d}E_a \mathrm{d}E_b \end{aligned} \tag{24}$$

where E_a and E_b are the energies of the decay pions a and b in the ω rest frame (see Appendix B of Ref. [2]). The mass-dependent width is then given by

$$\Gamma(w) \sim \int \sum_M |A|^2 \left(\frac{q}{w}\right) \mathrm{d}\Omega\,(w'\mathrm{d}w')\,\mathrm{d}\Omega'\,\mathrm{d}E_a\mathrm{d}E_b \tag{25}$$

where the sum on M, the initial z-component of the $b_1(1235)$ spin, is really meant to signify its average. One sees that the width formula (5) must be replaced by

$$\Gamma(w) \to \int w'^2 \mathrm{d}w'\,|\Delta_\omega(w')|^2 \left(\frac{q}{w}\right) \sum_\lambda |F^J_\lambda|^2 \int \mathrm{d}E_a\mathrm{d}E_b\,|\vec{a}\times\vec{b}|^2 \tag{26}$$

where $\Delta_\omega(w')$ is the Breit-Wigner form for the ω (one might as well set its width mass-independent). Note the factor w'^2 in the formula above; one factor comes from the phase space (24) and the other from (21).

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