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General formulation of covariant helicity-coupling amplitudes

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Abstract

A general formulation is given for constructing covariant helicity-coupling amplitudes involving two-body decays with arbitrary integer spins. The decay amplitudes are given exclusively in terms of both definite orbital angular momentum and total intrinsic spin. A systematic method is developed for calculating the energy and momentum dependence of daughter particles in the decay amplitudes, and a general formula for arbitrary integer spins is given. A number of illustrative examples is worked out, among which is that of the Higgs boson decay into two gauge bosons.

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I. INTRODUCTION

The purpose of this paper is to present a derivation of covariant helicity-coupling amplitudes for a parent state with an arbitrary integer spin J decaying into two daughter states with arbitrary integer spins s and σ. It was shown in a previous paper by the author [1] that, if a daughter particle has spin 1 or higher, the helicity-coupling amplitudes depend in general on the Lorentz factor $\gamma = E/m$, where m is the mass of the daughter and E is its energy in the parent rest frame. The paper emphasized a simplification that results from the exclusive use of spin tensors [2] and momenta defined along the helicity axis for the daughter states. This technique separates out the angular distribution contained in the D function from the problem of finding a proper energy and momentum dependence of the helicity-coupling amplitudes.

The author has recently written an updated preprint on the paper [3], which gives a more consistent formulation with detailed intermediate steps for calculating the amplitudes involving decays of practical importance. In this paper a different—and perhaps more efficient—technique has been developed for constructing the decay amplitudes. For a more basic exposition of the spin formalisms, the reader may wish to consult the CERN Yellow Report by the author [4] and also a recent paper by Filippini *et al.* [5] on covariant spin tensors.

The γ dependence is not unique, depending in general on the exact form of the decay amplitude one uses. It is shown that the functional form of γ becomes unique and simple, if the decay amplitudes are given in terms of definite orbital angular momentum ℓ and total intrinsic spin S. Therefore, one has systematically and exclusively utilized the projection operators corresponding to pure S and pure ℓ, with their definitions suitably extended in this paper to the relativistic case. This method provides, in addition, a means of systematically handling all the decays which involve photons in the final state on an equal footing.

Section II is devoted to an exposition of the classic decay amplitudes in the helicity formalism. What is new here is the general formula giving the number of independent helicity-coupling amplitudes for the decay process $J \to s + \sigma$, where the spins involved are any arbtrary integers. The results are also given in a tabular form for a few cases of practical importance. Sections III and IV cover the problem of constructing the decay amplitudes in the momentum space, the spin-1 wave functions and their projection operators. In particular, the form of a rank-J tensor is derived corresponding to the general wave function for an arbitrary integer spin J with a given z component of spin m, i.e., the tensor counterpart to the familiar ket state $|Jm\rangle$. To the best of the author's knowledge, such a tensor has been derived for the first time in a closed form. In section V, a derivation is given of the invariant ℓS-coupling amplitudes for the decay $J \to s + \sigma$ and, finally, the recoupling coefficient connecting them to those in the helicity basis is given—which represents the main result of this paper.

In sections VI through X, a wide-ranging and carefully chosen array of decay problems is given to illustrate the methods developed in this paper. The first example (Section VI) is the simplest which requires introduction of the Lorentz factor. A very important consequence is that the distribution resulting from an S-wave decay turns out to be anisotropic, which nevertheless tends toward an isotropic distribution in the nonrelativistic limit. The second example (Section VII) deals with a decay in which both the Lorentz factor and the dependence on the mass of the parent particle appear together in the decay amplitudes. In the third example (Section VIII), a polynomial dependence on the Lorentz factor appears for the first time. Moreover, this examples shows how different polynomials of the Lorentz factor could appear in the helicity-coupling amplitudes, depending on the way the tensors are used to construct them.

A decay mode in which both decay products have spins greater than one is treated in the fourth example (Section IX). Specifically, a hypothetical Higgs boson decay into two W bosons is considered, which includes the possibility of parity violation in the decay. It is shown in this example that the Lorentz factor is crucial in deducing that the Higgs boson coupling to two W bosons, in the high-mass limit, tends towards that of a boson decaying into two bosons (Goldstone bosons). In the final example (Section X), the case of a spin-1 object decaying into a spin-2 and a spin-1 particles is given. As the reader will discover, this example becomes very convoluted, requiring intrinsic spins 1, 2 and 3 and an orbital angular momentum of up to 4 in the final state. For such a case—and for more complex cases—it is very important that one is in posession of a general formula, obviating the need to work out contractions involving high-rank tensors (see Section V for such a general formula). If the spin-1 decay product turns out to be a photon, then this example illustrates a case in which the number of independent parameters in the helicity basis becomes smaller than that in the ℓ-S basis. For further examples involving photons, the reader may consult refs. [1] and [3].

Conclusions are given in Section XI.

II. HELICITY-COUPLING AMPLITUDES

Consider a state with spin(parity)=$J(\eta_J)$ decaying into two states with $s(\eta_s)$ and $\sigma(\eta_\sigma)$. The decay amplitudes are given, in the rest frame of J,

$$\begin{aligned} \mathcal{M}^J_{\lambda\nu}(\vartheta,\varphi;M) &\propto \langle \vartheta,\varphi,\lambda\nu|JM\lambda\nu\rangle\langle JM\lambda\nu|\ \mathcal{M}\ |JM\rangle \\ &\propto D^{J*}_{M\delta}(\varphi,\vartheta,0)F^J_{\lambda\nu} \end{aligned} \tag{1}$$

where $\mathcal{M}$ is the invariant operator for the decay, and λ and ν are the helicities of the two final-state particles 's' and 'σ' with $\delta = \lambda - \nu$. The symbol M stands for the z component of the spin J in a coordinate system fixed by production process. The helicities λ and ν are rotational invariants by definition. The direction of the break-up momentum of the decaying particle s is given by the angles ϑ and φ in the J rest frame. Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be the coordinate system fixed in the J rest frame. It is important to recognize, for applications to sequential decays, the exact nature of the body-fixed (helicity) coordinate system implied by the arguments of the D function given above. Let $\hat{x}_h$, $\hat{y}_h$ and $\hat{z}_h$ be the helicity coordinate system fixed by the s decay. Then by definition $\hat{z}_h$ describes the direction of the s in the J rest frame (termed the helicity axis in this paper) and the y axis is given by $\hat{y}_h \propto \hat{z} \times \hat{z}_h$ and $\hat{x}_h = \hat{y}_h \times \hat{z}_h$.

The helicity-coupling amplitude F^J given by

$$F^J_{\lambda\nu} \;\propto\; \langle JM\lambda\nu|\; \mathcal{M}\; |JM\rangle \tag{2}$$

is a rotational invariant. Parity conservation in the decay leads to the relationship

$$F^J_{\lambda\nu} = \eta_{_J}\eta_s\eta_\sigma(-)^{J-s-\sigma} F^J_{-\lambda-\nu} \tag{3}$$

while, if the decay products 's' and 'σ' are identical, the following additional relationship holds

$$F^J_{\lambda\nu} = (-)^J F^J_{\nu\lambda} \tag{4}$$

for both integer and half-integer spins.

The helicity-coupling amplitudes F^J are, in the nonrelativistic limit, related to the ℓS-coupling amplitudes $G^J_{\ell S}$ via

$$F^J_{\lambda\nu} = \sum_{\ell S}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0\, S\delta|J\delta)(s\lambda\, \sigma{-}\nu|S\delta)\; G^J_{\ell S} \tag{5}$$

where the coupling amplitudes have been given the normalization

$$\sum_{\ell S}|G^J_{\ell S}|^2 = \sum_{\lambda\nu}|F^J_{\lambda\nu}|^2 \tag{6}$$

and the $(\ell_1 m_1\, \ell_2 m_2|\ell_3 m_3)$ stands for the usual Clebsch-Gordan coefficients. The formula (5) for the helicity-coupling amplitudes results from the usual scheme of coupling of the angular momenta but with the z axis chosen along the helicity axis. Note that the orbital angular momentum ℓ has zero z component in this case and the particle 'σ' has z component $-\nu$. The formula (5) was given for the nonrelativistic case by Jacob and Wick [6] in Appendix B of their pioneering paper on helicity formalism. The main purpose of this paper is to show how this formula could be modified in the relativistic limit; the new formula is given in Section V.

It should be useful to give here a general formula for the number of independent amplitudes for $F^J_{\lambda\nu}$. From (1) one sees that the helicities are restricted by $|\lambda-\nu| \le J$. As there is a one-to-one correspondence between the number for independent $F^J_{\lambda\nu}$'s and that of $G^J_{\ell S}$'s if the particles involved are massive, the formula applies to both. It turns out that the formula is simpler if it is given as a sum of those for both positive and negative intrinsic parities of the parent particle. The combined number may be succinctly written

$$N_J = (a+b+1)(a-s+\sigma+1)+(s+\sigma-a)(2J+1) \tag{7}$$

where

$$a = \min\{J, s+\sigma\} \tag{8}$$
$$b = \min\{J, s-\sigma\}$$

and one has assumed here that $s \geq \sigma$. This formula breaks down into three cases as follows: If $J \geq s+\sigma$, one finds

$$N_J = (2s+1)(2\sigma+1) \tag{9}$$

But if $s-\sigma < J < s+\sigma$, one has

$$N_J = (J+s-\sigma+1)(J-s+\sigma+1)+(s+\sigma-J)(2J+1) \tag{10}$$

Finally, if $J < s-\sigma$, one obtains

$$N_J = (2\sigma+1)(2J+1) \tag{11}$$

The formula (9) is obvious from the form of the amplitude $F^J_{\lambda\nu}$ which has two subcripts corresponding to spins s and σ. The expression (11) shows that N_J is simply $2\sigma+1$ if $J=0$. In the ℓS-coupling scheme, the number of independent $G^J_{\ell S}$'s is merely given by $2\sigma+1$; this is the number of total intrinsic spin S, and ℓ must be equal to S if $J=0$. Finally, the number of independent amplitudes for a given intrinsic parity of the parent particle is given by $N^{(+)}_J = (N_J+1)/2$ if F^J_{00} is nonzero [see (3)], while the number is $N^{(-)}_J = (N_J-1)/2$ for the opposite intrinsic parity for which $F^J_{00}=0$. Note that N_J is always odd. If the two daughter particles are identical, then there exist additional constraints on the amplitudes and the resulting number N_J is smaller than that given above. The number N_J is tabulated in Table I for a few low values of the spins.

Table I.   Number of Independent Amplitudes

s	σ	J	$N_J^{(-)}$	$N_J^{(+)}$	N_J
0	0	0	0	1	1
1	0	0	0	1	1
1	0	1	1	2	3
1	0	2	1	2	3
1	1	0	1	2	3
1	1	1	3	4	7
1	1	2	4	5	9
1	1	3	4	5	9
2	0	0	0	1	1
2	0	1	1	2	3
2	0	2	2	3	5
2	0	3	2	3	5
2	1	0	1	2	3
2	1	1	4	5	9
2	1	2	6	7	13
2	1	3	7	8	15
2	1	4	7	8	15

III. DECAY AMPLITUDES IN MOMENTUM SPACE

The decay amplitude (1) is simply given by the helicity-coupling amplitude itself if one sets $\vartheta = \varphi = 0$. It is obvious now that the helicity-coupling amplitudes can be derived from the tensor formalism by restricting to the four-vectors defined along the z axis. Let p, q and k be the four-momenta for the states 'J', 's' and 'σ' with masses W, m and μ

$$p^\alpha = (p_0, \boldsymbol{p}), \quad p^2 = W^2, \quad q^\alpha = (q_0, \boldsymbol{q}), \quad q^2 = m^2, \quad k^\alpha = (k_0, \boldsymbol{k}), \quad k^2 = \mu^2 \tag{12}$$

and let $r = q - k$ be the break-up four-momentum. Using the Lorentz metric $\bar{g}_{\alpha\beta}$, one has

$$p_\alpha = \bar{g}_{\alpha\beta} p^\beta = (p_0, -\boldsymbol{p}\ ) \tag{13}$$

and similarly for the other four-vectors. In this paper, one has adopted the notations p, q, k and r to stand for *both* the four-momenta and the magnitudes of the 3-momenta. One can then define the following unitless quantities derived from them:

$$\gamma_s = \frac{q_0}{m}, \quad \gamma_s\beta_s = \frac{q}{m}, \quad \gamma_\sigma = \frac{k_0}{\mu} \quad \text{and} \quad \gamma_\sigma\beta_\sigma = \frac{k}{\mu} \tag{14}$$

One may now write an explicitly covariant expression (Lorentz scalar) for the helicity-coupling amplitudes

$$F^J_{\lambda\nu} = \sum_\alpha g_\alpha A_\alpha(\lambda\nu) \tag{15}$$

where

$$A_\alpha(\lambda\nu) = \left[p^n, r^\ell, \omega(\lambda), \varepsilon(-\nu), \phi^*(\delta)\right] \tag{16}$$

The square bracket here indicates that a Lorentz invariant amplitude is to be constructed out of the five variables: p, r, ω, ε and ϕ^*. As the momenta involved are all parallel with the helicity axis, this formula merely gives the energy and momentum dependence of the helicity-coupling amplitudes but no angular dependence, as this is already contained in the D function in the expression (1). The variables α stand for the set $\{\ell, S\}$, and the constants g_α are the analogue of the $G^J_{\ell S}$ in (5).

The covariant function A_α depends on p and r as well as the momentum-space wave functions (or tensors) $\phi^*(\delta)$, $\omega(\lambda)$ and $\varepsilon(-\nu)$ for the particles 'J', 's' and 'σ', where δ, λ and $-\nu$ are the z components of spin as defined before. Note that the complex conjugate of the J wave function appears in the above formula: it represents the initial state while those of 's' and 'σ' correspond to the final states. As shown with examples in later sections, one may set $n = 1$ or $n = 0$ without loss of generality, depending on the intrinsic parities involved. In other words, the four-vector p is used in the covariant amplitudes at most once, if necessary, in order to satisfy the requirement of parity conservation. The covariant function A_α can depend on any multiples (up to ℓ) of r, reflecting orbital angular momenta allowed in the decay.

A summary of notations used in this paper is given in Table II.

Table II. Two-body decay: $J \to s + \sigma$

	Parent	Daughter 1	Daughter 2
Spin	J	s	σ
Parity	η_J	η_s	η_σ
Helicity		λ	ν
Momentum	p	q	k
Energy	p_0	q_0	k_0
Mass	W	m	μ
Energy/Mass		γ_s	γ_σ
Velocity		β_s	β_σ
Wave function	$\phi^*(\lambda-\nu)$	$\omega(\lambda)$	$\varepsilon(-\nu)$

IV. WAVE FUNCTIONS AND PROJECTION OPERATORS

The polarization four-vectors or wave functions appropriate for the particles $J = 1$, '$s = 1$' and '$\sigma = 1$' are well known. Along with the relevant momenta,

$$\begin{aligned}
p^\alpha &= (W;\ 0,\ 0,\ 0) \\
q^\alpha &= (q_0;\ 0,\ 0,\ q) \\
&= (\gamma_s m;\ 0,\ 0,\ \gamma_s\beta_s m) \\
k^\alpha &= (k_0;\ 0,\ 0,\ -q) \\
&= (\gamma_\sigma \mu;\ 0,\ 0,\ -\gamma_\sigma\beta_\sigma \mu) \\
r^\alpha &= (q_0 - k_0;\ 0,\ 0,\ 2q)
\end{aligned} \tag{17}$$

where $W = q_0 + k_0$, $q_0 = \sqrt{m^2+q^2}$, $k_0 = \sqrt{\mu^2+q^2}$ and $r = q - k$, the wave functions in the J rest frame are given by

$$\begin{aligned}
\phi^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} (0;\ 1,\ \pm i,\ 0) \\
\phi^\alpha(0) &= (0;\ 0,\ 0,\ 1) \\
\omega^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} (0;\ 1,\ \pm i,\ 0) \\
\omega^\alpha(0) &= (\gamma_s\beta_s;\ 0,\ 0,\ \gamma_s) \\
\varepsilon^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} (0;\ 1,\ \pm i,\ 0) \\
\varepsilon^\alpha(0) &= (-\gamma_\sigma\beta_\sigma;\ 0,\ 0,\ \gamma_\sigma)
\end{aligned} \tag{18}$$

Note that

$$p_\alpha \phi^\alpha(\lambda) = q_\alpha \omega^\alpha(\lambda) = k_\alpha \varepsilon^\alpha(\lambda) = 0$$

for any λ.

These polarization four-vectors satisfy

$$\begin{aligned}
p_\alpha \phi^\alpha(m) &= 0 \\
\phi^*_\alpha(m)\phi^\alpha(m') &= -\delta_{mm'} \\
\sum_m \phi_\alpha(m)\phi^*_\beta(m) &= \tilde{g}_{\alpha\beta}(W)
\end{aligned} \tag{19}$$

where

$$\tilde{g}_{\alpha\beta}(W) = -\bar{g}_{\alpha\beta} + \frac{p_\alpha p_\beta}{W^2} \tag{20}$$

The last equation of (19) is the usual projection operator for spin-1 states. Note that, in the J rest frame, $\tilde{g}(W)$ has a zero time-component and $+1$ for the space-components. ω and ε satisfy similar conditions, but with their own $\tilde{g}$'s, i.e., $\tilde{g}(m)$ and $\tilde{g}(\mu)$.

One is now ready to exhibit all the Lorentz invariants involving spin-1 wave functions. One has adopted, in this paper, exclusive use of the modified Lorentz metric $\tilde{g}(W)$ for all the Lorentz scalars in the problem:

$$[a \cdot b] \equiv a^\alpha \, \tilde{g}_{\alpha\beta}(W) \, b^\beta = (\boldsymbol{a} \cdot \boldsymbol{b}) \tag{21}$$

where a and b are arbitrary four-vectors. $\boldsymbol{a}$ and $\boldsymbol{b}$ are three-vectors defined in the J rest frame. The rationale for this approach is that pure-spin projection operators should be used to form Lorentz scalars, since the wave functions for s and σ are not those of a pure spin-1 state in the J rest frame (a general formulation of this approach is given in the next section). Inspection of (19) shows that the modified metric $\tilde{g}(W)$ is in fact equal to a spin-1 projection operator consisting of a new spin-1 wave function, e.g. $\chi(m)$, defined to be the same as $\phi(m)$ [see (18)]. One difference is that $\phi(m)$ is a wave function defined in the initial system and $\chi(m)$ is that set up in the final state. Note that

$$[a \cdot b] = \sum_m \; a^\mu \, \chi^*_\mu(m) \; b^\nu \, \chi_\nu(m) = \sum_m \; a^\mu \, \chi_\mu(m) \; b^\nu \, \chi^*_\nu(m) \tag{22}$$

If the quantization axis for $\chi(m)$ is defined along the helicity axis and if a and/or b are either the wave functions defined with the same quantization axis or four-vectors with zero x and y components, then an important simplification occurs: the sum on m disappears in (22).

Using the prescription (21) or (22), it can be shown that all the Lorentz scalars evaluated in the J rest frame may be written as

$$\begin{aligned} [r \cdot \omega(m)] &= \gamma_s \, r \, \delta_{m\,0} \\ [r \cdot \varepsilon(m)] &= \gamma_\sigma \, r \, \delta_{m\,0} \end{aligned}$$

$$
\begin{aligned}
[r \cdot \phi^*(m)] &= r\, \delta_{m\,0} \\
[\omega(m) \cdot \varepsilon(m')] &= (-1)^m \left[m^2 + \gamma_s\gamma_\sigma(1-m^2)\right] \delta_{m,\,-m'} \\
[\omega(m) \cdot \phi^*(m')] &= \left[m^2 + \gamma_s(1-m^2)\right] \delta_{m,\,m'} \\
[\varepsilon(m) \cdot \phi^*(m')] &= \left[m^2 + \gamma_\sigma(1-m^2)\right] \delta_{m,\,m'}
\end{aligned}
\tag{23}
$$

There exists a second form of Lorentz invariant involving the totally antisymmetric rank-4 tensor. For any four vectors a, b, c and d, it can be written

$$
[a\,b\,c\,d] = \epsilon_{\alpha\beta\gamma\delta}\; a^\alpha\, b^\beta\, c^\gamma\, d^\delta \tag{24}
$$

Relevant invariants in the problem are

$$
\begin{aligned}
[p\,\omega(m)\,r\,\phi^*(m')] &= imWr\,\delta_{m\,m'} \\
[p\,\varepsilon(m)\,r\,\phi^*(m')] &= imWr\,\delta_{m\,m'} \\
[p\,\omega(m)\,r\,\varepsilon(m')] &= -imWr\,\delta_{m\,-m'} \\
[p\,\omega(m)\,\varepsilon(m')\,\phi^*(m'')] &= iW\, \left[m(1-m''^2) + m''(1-m'^2)\gamma_\sigma - m'(1-m^2)\gamma_s\right] \delta_{m'',\,m+m'}
\end{aligned}
\tag{25}
$$

The spin-2 wave functions can be written

$$
\phi^{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1\, 1m_2|2m)\phi^\alpha(m_1)\phi^\beta(m_2) \tag{26}
$$

where $m = m_1 + m_2$. This is orthogonal to p, symmetric in the two indices and traceless under contraction with g or $\tilde{g}(W)$. The spin-2 wave functions for s and σ are constructed in the same way, but they are not traceless with respect to the modified metric $\tilde{g}(W)$. For example, note that, for $s = 2$,

$$
[\tilde{g}(W) : \omega(m)] = \sqrt{\frac{2}{3}}(\gamma_s^2 - 1)\delta_{m\,0} \tag{27}
$$

where a colon indicates contraction over two neighboring indices. It becomes traceless in the limit $\gamma_s \to 1$. The spin-3 wave functions can be constructed in a similar manner, starting with spin-2 and spin-1 wave functions, as follows:

$$
\begin{aligned}
\phi^{\alpha\beta\gamma}(m) &= \sum_{n_1 m_2} (2n_1\, 1m_3|3m)\phi^{\alpha\beta}(n_1)\phi^\gamma(m_3) \\
&= \sum_{m_1 m_2 m_3} (1m_1 1m_2|2n_1)(2n_1 1m_3|3m)\phi^\alpha(m_1)\phi^\beta(m_2)\phi^\gamma(m_3)
\end{aligned}
\tag{28}
$$

where $m = m_1 + m_2 + m_3$. Note that these are orthogonal to p and are symmetric under interchange of any pairs in $\{\alpha, \beta, \gamma\}$. They are also traceless, i.e., they vanish under contraction with g or $\tilde{g}(W)$ for any pairs in $\{\alpha, \beta, \gamma\}$. In general, the wave function for a particle of spin-J is a rank-J tensor

$$\phi^{\alpha_1\alpha_2\cdots\alpha_J}(m) = \sum_{m_1 m_2 \cdots} (1m_1 1m_2|2n_1)(2n_1 1m_3|3n_2)\cdots(J{-}1\, n_{J-2}\ 1m_{J}|Jm) \\ \phi^{\alpha_1}(m_1)\phi^{\alpha_2}(m_2)\cdots\phi^{\alpha_J}(m_{J}) \tag{29}$$

with $m = m_1 + m_2 + \cdots + m_{J}$ and normalized by

$$\phi^*_{\alpha\beta\gamma\cdots}(m)\phi^{\alpha\beta\gamma\cdots}(m') = (-)^J \delta_{mm'} \tag{30}$$

and

$$[\phi^*(m) \otimes \phi(m')] = \delta_{mm'} \tag{31}$$

where the symbol $\otimes$ stands for contraction of two equal-rank tensors with the modified metric $\tilde{g}(W)$.

The Clebsch-Gordan coefficients appearing in (29) have the following simple expressions [7]:

$$\begin{aligned} (j\, m{+}1\ 1\ {-1}|j{+}1\, m) &= \left[\frac{(j-m)(j-m+1)}{(2j+1)(2j+2)}\right]^{\frac{1}{2}} \\ (j\, m\ 1\, 0|j{+}1\, m) &= \left[\frac{(j-m+1)(j+m+1)}{(2j+1)(j+1)}\right]^{\frac{1}{2}} \\ (j\, m{-}1\ 1\ {+1}|j{+}1\, m) &= \left[\frac{(j+m)(j+m+1)}{(2j+1)(2j+2)}\right]^{\frac{1}{2}} \end{aligned} \tag{32}$$

Using these formulas, one deduces that the general spin-J wave function (29) can be transformed into

$$\phi^{\delta_1\cdots\delta_J}(m) = [a^J(m)]^{\frac{1}{2}} \sum_{m_0} 2^{m_0/2} \sum_{P} \phi^{\alpha_1}(+)\cdots\phi^{\beta_1}(0)\cdots\phi^{\gamma_1}(-)\cdots \tag{33}$$

where

$$a^J(m) = \frac{(J+m)!(J-m)!}{(2J)!} \tag{34}$$

and the indices $\{\delta_1 \cdots \delta_J\}$ have been broken up into three distinct sets in the second summation, i.e., $\{\alpha_i\}$ with $(i = 1, m_+)$, $\{\beta_i\}$ with $(i = 1, m_0)$ and $\{\gamma_i\}$ with $(i = 1, m_-)$, where $m_\pm$ stands for the numbers of $\phi(\pm)$'s and m_0 for $\phi(0)$'s. Note that

$$J = m_+ + m_0 + m_- \quad \text{and} \quad m = m_+ - m_- \tag{35}$$

and that

$$2m_\pm = J \pm m - m_0 \tag{36}$$

It is apparent that the right-hand side must be always even. The first sum in (33) goes over the allowed values of m_0 given J and m. It is clear that the maximum is given by $J - m$, so that m_0 ranges from 0(1), 2(3),$\cdots$, to $J - m$ =even(odd). The second sum in (33) represents a summation on the permutations

$$\{(+)(+)\cdots(0)(0)\cdots(-)(-)\cdots\}$$

It is seen readily that the number of terms in the summation is

$$b^J(m, m_0) = \frac{J!}{m_+! m_0! m_-!} \tag{37}$$

Note the following useful relationship:

$$\phi(-m) = (-)^m \phi^*(m) \tag{38}$$

It is best to illustrate these formulas with examples for $J = 1$, 2 and 3. For $J = 1$ one finds that (33) reduces to identities for $\phi(+)$ and $\phi(0)$. For $J = 2$, one finds

$$\begin{aligned}
\phi^{\alpha\beta}(+2) &= \phi^\alpha(+)\,\phi^\beta(+) \\
\phi^{\alpha\beta}(+1) &= \frac{1}{\sqrt{2}}\Big[\phi^\alpha(+)\,\phi^\beta(0) + \phi^\alpha(0)\,\phi^\beta(+)\Big] \\
\phi^{\alpha\beta}(0) &= \frac{1}{\sqrt{6}}\Big[\phi^\alpha(+)\,\phi^\beta(-) + \phi^\alpha(-)\,\phi^\beta(+)\Big] + \sqrt{\frac{2}{3}}\phi^\alpha(0)\,\phi^\beta(0)
\end{aligned} \tag{39}$$

and, for $J = 3$, the wave functions take on the form

$$
\begin{aligned}
\phi^{\alpha\beta\gamma}(+3) &= \phi^{\alpha}(+)\,\phi^{\beta}(+)\,\phi^{\gamma}(+) \\
\phi^{\alpha\beta\gamma}(+2) &= \frac{1}{\sqrt{3}}\Big[\phi^{\alpha}(+)\,\phi^{\beta}(+)\,\phi^{\gamma}(0) + \phi^{\alpha}(+)\,\phi^{\beta}(0)\,\phi^{\gamma}(+) + \phi^{\alpha}(0)\,\phi^{\beta}(+)\,\phi^{\gamma}(+)\Big] \\
\phi^{\alpha\beta\gamma}(+1) &= \frac{1}{\sqrt{15}}\Big[\phi^{\alpha}(+)\,\phi^{\beta}(+)\,\phi^{\gamma}(-) + \phi^{\alpha}(+)\,\phi^{\beta}(-)\,\phi^{\gamma}(+) + \phi^{\alpha}(-)\,\phi^{\beta}(+)\,\phi^{\gamma}(+)\Big] \\
&\quad + \frac{2}{\sqrt{15}}\Big[\phi^{\alpha}(+)\,\phi^{\beta}(0)\,\phi^{\gamma}(0) + \phi^{\alpha}(0)\,\phi^{\beta}(+)\,\phi^{\gamma}(0) + \phi^{\alpha}(0)\,\phi^{\beta}(0)\,\phi^{\gamma}(+)\Big] \\
\phi^{\alpha\beta\gamma}(0) &= \frac{1}{\sqrt{10}}\Big[\phi^{\alpha}(0)\,\phi^{\beta}(+)\,\phi^{\gamma}(-) + \phi^{\alpha}(0)\,\phi^{\beta}(-)\,\phi^{\gamma}(+) + \phi^{\alpha}(-)\,\phi^{\beta}(+)\,\phi^{\gamma}(0) \\
&\quad + \phi^{\alpha}(+)\,\phi^{\beta}(-)\,\phi^{\gamma}(0) + \phi^{\alpha}(+)\,\phi^{\beta}(0)\,\phi^{\gamma}(-) + \phi^{\alpha}(-)\,\phi^{\beta}(0)\,\phi^{\gamma}(+)\Big] \\
&\quad + \sqrt{\frac{2}{5}}\phi^{\alpha}(0)\,\phi^{\beta}(0)\,\phi^{\gamma}(0)
\end{aligned}
\tag{40}
$$

V. INVARIANT ℓS-COUPLING AMPLITUDES

In order to find a connection of the tensor formalism with that of the ℓS-coupling scheme, one needs to develop the concept of total intrinsic spin S formed out of the 's' and 'σ' polarization four-vectors and that of the pure orbital angular momentum ℓ built out of r. Consider now a wave function $\chi^S(m_s)$ which is to form the basis for constructing the ket state $|Sm_s\rangle$. One demands that this wave function have zero time component in the J rest frame very similar to ϕ [see (18) and (29)].

$$
\chi^S_{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_\sigma}(m_s) = \sum_{m_a m_b} (sm_a\,\sigma m_b|Sm_s)\chi^s_{\alpha_1\cdots\alpha_s}(m_a)\chi^\sigma_{\beta_1\cdots\beta_\sigma}(m_b) \tag{41}
$$

where rank-s tensor $\chi^s(m_a)$ and rank-σ tensor $\chi^\sigma(m_b)$ are the 'rest-state'—and fictitious—wave functions, invented for constructing projection operators, and hence exactly equal to (33) with the index J changed to either s or σ. Consequently, χ^S is a tensor of rank $s+\sigma$ which acts on rank-s ω and rank-σ ε tensors, which are the 'correct' relativistic tensors corresponding to the decay products s and σ. The corresponding projection operator is a tensor of rank $2(s+\sigma)$ given by

$$
P^S = \sum_{m_s} \chi^S(m_s)\chi^{S\,*}(m_s) \tag{42}
$$

It is seen that the invariant amplitudes must contain the products

$$\begin{aligned} \left[\chi^{s\,*}(m_a) \otimes \omega(\lambda)\right] &= f^s_\lambda(\gamma_s)\ \delta_{m_a\,\lambda} \\ \left[\chi^{\sigma\,*}(m_b) \otimes \varepsilon(-\nu)\right] &= f^\sigma_\nu(\gamma_\sigma)\ \delta_{m_b\,-\nu} \end{aligned} \tag{43}$$

where the symbol $\otimes$ indicates, once again, a contraction between two tensors of equal rank with the modified metric $\tilde{g}(W)$. The functions f are normalized to 1 as $\gamma \to 1$ and are given below. The invariant amplitude corresponding to a state of pure spin S is then

$$e^S(\lambda\nu) = \left[\omega(\lambda) \otimes \chi^{S\,*}(m_s) \otimes \varepsilon(-\nu)\right] = (s\lambda\ \sigma{-}\nu|S\delta) f^s_\lambda(\gamma_s)\ f^\sigma_\nu(\gamma_\sigma)\ \delta_{m_s\delta} \tag{44}$$

where $\delta = \lambda - \nu$. The first $\otimes$ signifies a contraction of the indices $\{\alpha_1 \cdots \alpha_s\}$ [see (41)] with the modified Lorentz metric $\tilde{g}(W)$, and the second one of the indices $\{\beta_1 \cdots \beta_\sigma\}$. A very important simplification results from using the specialized wave functions defined along the helicity axis: there is no summation on the index m_s in the projection operator P^S. The projection operator is merely given by $\chi^S(\delta)$ multiplied by (44).

The function $f^s_\lambda(\gamma_s)$ for $s = 1$ is trivially given by, from (23),

$$\begin{aligned} \xi_s(\lambda) \equiv f^{(1)}_\lambda(\gamma_s) &= \left[\chi^{(1)*}(\lambda) \cdot \omega(\lambda)\right] \\ &= 1 \quad \text{for} \quad \lambda = \pm 1 \\ &= \gamma_s \quad \text{for} \quad \lambda = 0 \end{aligned} \tag{45}$$

where a dot within the bracket indicates, once again, a contraction of two four-vectors with the modified Lorentz metric $\tilde{g}(W)$. For s=2, the function f takes on the form

$$f^s_\lambda(\gamma_s) = \sum_{\lambda_1\lambda_2} (1\lambda_1\ 1\lambda_2|2\lambda)^2 \xi_s(\lambda_1)\xi_s(\lambda_2) \tag{46}$$

so that

$$\begin{aligned} f^{(2)}_\lambda(\gamma_s) &= 1 \quad \text{for} \quad \lambda = \pm 2 \\ &= \gamma_s \quad \text{for} \quad \lambda = \pm 1 \\ &= \frac{2}{3}\gamma_s^2 + \frac{1}{3} \quad \text{for} \quad \lambda = 0 \end{aligned} \tag{47}$$

Similarly for $s = 3$, one finds

$$f^s_\lambda(\gamma_s) = \sum_{\lambda_1\lambda_2\lambda_3} (1\lambda_1\, 1\lambda_2|2\lambda_a)^2 (2\lambda_a\, 1\lambda_3|3\lambda)^2 \xi_s(\lambda_1)\xi_s(\lambda_2)\xi_s(\lambda_3) \tag{48}$$

so that

$$\begin{aligned} f^{(3)}_\lambda(\gamma_s) &= 1 \quad \text{for} \quad \lambda = \pm 3 \\ &= \gamma_s \quad \text{for} \quad \lambda = \pm 2 \\ &= \frac{4}{5}\gamma_s^2 + \frac{1}{5} \quad \text{for} \quad \lambda = \pm 1 \\ &= \frac{2}{5}\gamma_s^3 + \frac{3}{5}\gamma_s \quad \text{for} \quad \lambda = 0 \end{aligned} \tag{49}$$

Finally, one may work out the function f for $s = 4$ as well: from

$$f^s_\lambda(\gamma_s) = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} (1\lambda_1\, 1\lambda_2|2\lambda_a)^2 (2\lambda_a\, 1\lambda_3|3\lambda_b)^2 (3\lambda_b\, 1\lambda_4|4\lambda)^2 \xi_s(\lambda_1)\xi_s(\lambda_2)\xi_s(\lambda_3)\xi_s(\lambda_4) \tag{50}$$

one sees that

$$\begin{aligned} f^{(4)}_\lambda(\gamma_s) &= 1 \quad \text{for} \quad \lambda = \pm 4 \\ &= \gamma_s \quad \text{for} \quad \lambda = \pm 3 \\ &= \frac{6}{7}\gamma_s^2 + \frac{1}{7} \quad \text{for} \quad \lambda = \pm 2 \\ &= \frac{4}{7}\gamma_s^3 + \frac{3}{7}\gamma_s \quad \text{for} \quad \lambda = \pm 1 \\ &= \frac{8}{35}\gamma_s^4 + \frac{24}{35}\gamma_s^2 + \frac{3}{35} \quad \text{for} \quad \lambda = 0 \end{aligned} \tag{51}$$

The general formula for the f-functions can be obtained by inspecting (43) and (33):

$$f^j_m(\gamma) = a^j(m) \sum_{m_0} b^j(m, m_0)\, (2\gamma)^{m_0} \tag{52}$$

where j, m and γ can stand for s, λ and γ_s or σ, ν and γ_σ. As in the previous section, m_0 ranges from 0(1), 2(3),$\cdots$, to $J - m$ =even(odd). It is easy to verify that this formula gives the results (45) for $j = 1$, (47) for $j = 2$, (49) for $j = 3$ and (51) for $j = 4$.

The analogue of the ket state $|\ell m\rangle$ may be represented by a rank-ℓ tensor $\tau^\ell(m)$, defined to have zero time component in the J rest frame, since the orbital angular momentum is defined only in this frame. The tensor $\tau^\ell(m)$, which corresponds to the 'rest-state'—and fictious—wave function invented for constructing projection operators, is analogous to the tensors $\chi^s(m_a)$ and $\chi^\sigma(m_b)$ introduced in (41). Therefore, the tensor $\tau^\ell(m)$ is exactly equal to (33) with the index J changed to ℓ. The corresponding projection operator is a tensor of rank 2ℓ given by

$$P^\ell = \sum_m \tau^\ell(m)\tau^{\ell *}(m) \tag{53}$$

Again, one can simplify the treatment of orbital angular momentum by defining $\tau^\ell(m)$ along the helicity axis [see ϕ in (18)]. If $\ell = 1$, one finds

$$\left[\tau^{(1)*}(m) \cdot r\right] = r\, \delta_{m\,0} \tag{54}$$

This can be easily generalized, so that one finds

$$\left[\tau^{\ell *}(m) \otimes r\, r \cdots\right] = c_\ell\, r^\ell\, \delta_{m\,0} \tag{55}$$

where

$$\begin{aligned} c_\ell &= (1010|20)(2010|30)\cdots(\ell{-}1\,0\;10|\ell\,0) \\ &= \ell! \left[\frac{2^\ell}{(2\ell)!}\right]^{\frac{1}{2}} \end{aligned} \tag{56}$$

One is now ready to evaluate the final element of the invariant ℓS-coupling amplitudes:

$$\left[p^n, \chi^S(\delta), \tau^\ell(0), \phi^*(\delta)\right] \tag{57}$$

where $n = 1$ for $s + \sigma + \ell - J$ odd and $n = 0$ otherwise. For example, if $\sigma = 0$ and $s = \ell = J = 1$, then the invariant amplitude can be written as

$$[p\, \chi(\delta)\, \tau(0)\, \phi^*(\delta)] \propto W\,(10\,1\delta|1\delta) \tag{58}$$

The right-hand side results from evaluating the expression in the J rest frame. It can be shown that the last expression in (25) with $\gamma_s = \gamma_\sigma = 1$ is equivalent to the result above. Consider another example: if $\sigma = 0$, $\ell = 2$ and $s = J = 1$, then the invariant amplitude can be written, from (23) and (26),

$$[\chi(\delta) \cdot \tau^{(2)}(0) \cdot \phi^*(\delta)] \propto (20\,1\delta|1\delta) \tag{59}$$

One can infer in general that the invariant amplitude of (57) should take on the form, in the J rest frame,

$$\left[p^n, \chi^S(\delta), \tau^\ell(0), \phi^*(\delta)\right] \propto W^n\,(\ell 0\, S\delta|J\delta) \tag{60}$$

The left-hand side is in reality proportional to the matrix element

$$\langle \ell 0\, S\delta|\mathcal{M}|J\delta\rangle \tag{61}$$

One can expand the state $|J\delta\rangle$ in the usual way

$$|J\delta\rangle = \sum_{j_1\delta_1 j_2\delta_2} (j_1\delta_1\, j_2\delta_2|J\delta)|j_1\delta_1\, j_2\delta_2\rangle \tag{62}$$

This is reduced to (60), if $\mathcal{M}$ is applied from the left first and then followed by the ket states $|\ell 0\rangle$ and $|S\delta\rangle$.

The invariant helicity-coupling amplitude may now be written, from (44) and (60),

$$F^J_{\lambda\nu} = \sum_{\ell S} g_{\ell\, S}\ A_{\ell S}(\lambda\nu) \tag{63}$$

where

$$A_{\ell S}(\lambda\nu) = \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0\, S\delta|J\delta)(s\lambda\, \sigma{-}\nu|S\delta)\ W^n\, r^\ell\, f^s_\lambda(\gamma_s)\ f^\sigma_\nu(\gamma_\sigma) \tag{64}$$

where the square-root factor has been introduced so that the formula above has an appearance similar to (5). The coefficient c_ℓ has been absorbed into g. The complex parameters g are unknown, to be determined from experiment. Once again, it is to be noted that $n = 1$ for $s + \sigma + \ell - J$ odd and $n = 0$ otherwise. One could make the right-hand side unitless by substituting W and and r by $\widehat{W} = W/W_0$ and $\widehat{r} = r/r_0$ where r_0 refers to the r corresponding to the nominal mass values W_0, m_0 and μ_0. One sees then that (63) reduces to (5), i.e., $g_{\ell\, S} \to G^J_{\ell S}$ in the limit $\widehat{W} \to 1$, $\widehat{r} \to 1$, $f^s_\lambda \to 1$ and $f^\sigma_\nu \to 1$.

The expression (63) is the main result of this paper. The r^ℓ dependence is familiar, but the W and γ dependence are not; the W factor is necessary to insure Lorentz covariance in four dimensions, and the functional forms on γ result from the boosted wave functions one needs to employ for s and σ.

A few examples are given below for illustration.

VI. $b_1(1235) \to \omega + \pi$

Let 'J', 's' and 'σ' stand for the $b_1(1235)$, the ω and the π. The net intrinsic parity is given by $\eta_J \eta_s \eta_\sigma = +1$ and $F^J_\lambda = +F^J_{-\lambda}$, and there are two allowed orbital angular momenta, i.e., $\ell = 0$ or $\ell = 2$. The helicity-coupling amplitudes have the following expansion in the nonrelativistic limit (see (5)):

$$\begin{aligned} \sqrt{2} F^J_+ &= \sqrt{\frac{2}{3}} G^J_0 + \sqrt{\frac{1}{3}} G^J_2 \\ F^J_0 &= \sqrt{\frac{1}{3}} G^J_0 - \sqrt{\frac{2}{3}} G^J_2 \end{aligned} \tag{65}$$

where $J = 1$. According to the Particle Data Group [8], one has, experimentally,

$$\left| \frac{G^J_2}{G^J_0} \right| = 0.26 \pm 0.04 \tag{66}$$

There are two covariant decay amplitudes corresponding to S- and D-waves in the problem, before introduction of projection operators,

$$\begin{aligned} A_0(\lambda) &= [\omega(\lambda) \cdot \phi^*(\lambda)] \\ A_2(\lambda) &= [\omega(\lambda) \cdot \tau^{(2)}(0) \cdot \phi^*(\lambda)]\, c_2\, r^2 \end{aligned} \tag{67}$$

The form of the A_2 given above may be more efficient, especially for high values of ℓ, than that given in the earlier paper [1]:

$$A_2(\lambda) = [\omega(\lambda) \cdot r][r \cdot \phi^*(\lambda)] - \frac{1}{3} r^2 \, [\omega(\lambda) \cdot \phi^*(\lambda)]$$

The helicity-coupling amplitudes are given by

$$F^J_\lambda = g_0 A_0(\lambda) + g_2 A_2(\lambda) \tag{68}$$

where g_0 and g_2 are arbitrary constants. Evaluating A's in the J rest frame, one obtains

$$\begin{aligned} F^J_+ &= g_0 - \frac{1}{3} g_2 r^2 \\ F^J_0 &= \gamma_s \left(g_0 + \frac{2}{3} g_2 r^2 \right) \end{aligned} \tag{69}$$

where $J = 1$. In the limit $\gamma_s \to 1$, the expressions of (69) reduce to those of (65) with the replacement

$$G_0^J = \sqrt{3}\, g_0, \qquad G_2^J = -\sqrt{\frac{2}{3}}\, g_2\, r^2 \tag{70}$$

When the amplitudes are constructed with the aid of projection operators, the invariant helicity-coupling amplitudes are simply given by (63):

$$\begin{aligned} \sqrt{2} F_+^J &= \sqrt{\frac{2}{3}} g_0 + \sqrt{\frac{1}{3}} g_2 r^2 \\ F_0^J &= \left(\sqrt{\frac{1}{3}} g_0 - \sqrt{\frac{2}{3}} g_2 r^2 \right) \gamma_s \end{aligned} \tag{71}$$

The g's in this expression are of course proportional to the g's in (69).

It is instructive to work out the angular distribution for this decay. Suppose that ω decays into 3π and the orientation of the normal to its decay plane is given by (ϑ', φ') in the helicity coordinate system $\{\hat{x}_h, \hat{y}_h, \hat{z}_h\}$ as defined in Section III. Then the overall decay amplitude is

$$\mathcal{M}^J(\vartheta, \varphi, \vartheta', \varphi', M) \propto \sum_\lambda D_{M\lambda}^{J*}(\varphi, \vartheta, 0) F_\lambda^J D_{\lambda 0}^{s*}(\varphi', \vartheta', 0) \tag{72}$$

In terms of the density matrix ρ defined in the J rest frame, the angular distribution I is

$$\begin{aligned} I(\vartheta, \varphi, \vartheta', \varphi') \propto \sum_{\substack{MM' \\ \lambda\lambda'}} & \rho_{MM'} D_{M\lambda}^{J*}(\varphi, \vartheta, 0) D_{M'\lambda'}^{J}(\varphi, \vartheta, 0) \\ & F_\lambda^J F_{\lambda'}^{J*} D_{\lambda 0}^{s*}(\varphi', \vartheta', 0) D_{\lambda' 0}^{s}(\varphi', \vartheta', 0) \end{aligned} \tag{73}$$

For the purpose of illustration, it is sufficient to take the special case in which $\rho_{00} = 1$ and all the other elements are zero. Then, after integrating over φ and φ', one finds

$$I(\vartheta, \vartheta') \propto \sum_\lambda \, [d_{0\lambda}^J(\vartheta)]^2 \; |F_\lambda^J|^2 \; [d_{\lambda 0}^s(\vartheta')]^2 \tag{74}$$

This leads to two very similar distributions

$$I(\vartheta) \propto |F_0^J|^2 \cos^2(\vartheta) + |F_+^J|^2 \sin^2(\vartheta) \tag{75}$$

$$I(\vartheta') \propto |F_0^J|^2 \cos^2(\vartheta') + |F_+^J|^2 \sin^2(\vartheta') \tag{76}$$

If one assumes that g_i's are relatively real, one obtains, from (69),

$$\begin{aligned} I(\vartheta) \propto\ & g_0^2[(\gamma_s^2-1)\cos^2(\vartheta)+1] \\ & +\frac{2}{3}g_0g_2\,r^2[(2\gamma_s^2+1)\cos^2(\vartheta)-1] \\ & +\frac{1}{9}g_2^2\,r^4[(4\gamma_s^2-1)\cos^2(\vartheta)+1] \end{aligned} \tag{77}$$

$$\begin{aligned} I(\vartheta') \propto\ & g_0^2[(\gamma_s^2-1)\cos^2(\vartheta')+1] \\ & +\frac{2}{3}g_0g_2\,r^2[(2\gamma_s^2+1)\cos^2(\vartheta')-1] \\ & +\frac{1}{9}g_2^2\,r^4[(4\gamma_s^2-1)\cos^2(\vartheta')+1] \end{aligned} \tag{78}$$

Two noteworthy results of this exercise are: (a)the S-wave term containing the factor g_0^2 is no longer isotropic in the cosines for $\gamma_s > 1$, and (b)the D-wave term with g_2^2 is a polynomial of order 2 in the cosines, reflecting the fact that the parent particle has $J = 1$. It is important to note, in addition, that the singularities implicit with the presence of cosines in the amplitudes are cancelled by r^2 and γ_s.

It is illuminating to work out the angular distribution again within the nonrelativistic formalism with canonical quantization. Integrating over the variables corresponding to the ω decay and over φ, one finds, for $\rho_{00} = 1$,

$$I(\vartheta) \propto \sum_{\ell\ell'} G_\ell^J G_{\ell'}^{J*} \sum_m (\ell m\ s\ {-m}|J0)(\ell' m\ s\ {-m}|J0)\ Y_\ell^m(\vartheta,0)Y_{\ell'}^m(\vartheta,0) \tag{79}$$

With the substitutions (70), it can be shown that this angular distribution reduces to that of (77) in the limit $\gamma_s \to 1$.

VII. $\bar{p}p(^3P_2) \to f_2(1270) + \pi$

The net intrinsic parity is $\eta_J\eta_s\eta_\sigma = -1$ and there are two helicity-coupling amplitudes $F_2^{(2)}$ and $F_1^{(2)}$ corresponding to $\ell = 1$ and $\ell = 3$:

$$\begin{aligned} \sqrt{2}F_2^{(2)} &= -\frac{2}{\sqrt{5}}G_1^{(2)} - \frac{1}{\sqrt{5}}G_3^{(2)} \\ \sqrt{2}F_1^{(2)} &= -\frac{1}{\sqrt{5}}G_1^{(2)} + \frac{2}{\sqrt{5}}G_3^{(2)} \end{aligned} \tag{80}$$

in the nonrelativistic limit.

The covariant amplitudes corresponding to pure orbital angular momenta are, before projection operators are introduced,

$$\begin{aligned} A_1 &= [p\ r\ \omega \cdot \phi^*] \\ A_3 &= [p\ \omega \cdot \tau^{(3)}(0) \cdot \phi^*]\, c_3\, r^3 \end{aligned} \tag{81}$$

The amplitudes with $\lambda = +2$ and $\lambda = +1$ lead to

$$\begin{aligned} F_2^{(2)} &= W \left(g_1 - \frac{1}{5} g_3\, r^2 \right) r \\ F_1^{(2)} &= W \gamma_s \left(\frac{1}{2} g_1 + \frac{2}{5} g_3\, r^2 \right) r \end{aligned} \tag{82}$$

for two arbitrary complex constants g_1 and g_3. With the technique of projection operators, the F's assume the form, from (63),

$$\begin{aligned} \sqrt{2} F_2^{(2)} &= -W \left(\frac{2}{\sqrt{5}} g_1 + \frac{1}{\sqrt{5}} g_3\, r^2 \right) r \\ \sqrt{2} F_1^{(2)} &= W \gamma_s \left(-\frac{1}{\sqrt{5}} g_1 + \frac{2}{\sqrt{5}} g_3\, r^2 \right) r \end{aligned} \tag{83}$$

With a proper redefinition of the g's, it can shown that (82) and (83) are identical.

VIII. $a_3(2050) \to f_2(1270)\pi$

This decay is so far unobserved, but it affords an opportunity to explore new territory regarding the structure of helicity-coupling amplitudes. There are three F^J's corresponding to ℓ =1, 3 and 5:

$$\begin{aligned} \sqrt{2} F_2^J &= \sqrt{\frac{2}{7}} G_1^J + \sqrt{\frac{2}{3}} G_3^J + \frac{1}{\sqrt{21}} G_5^J \\ \sqrt{2} F_1^J &= \frac{4}{\sqrt{35}} G_1^J - \frac{1}{\sqrt{15}} G_3^J - \sqrt{\frac{10}{21}} G_5^J \\ F_0^J &= \frac{3}{\sqrt{35}} G_1^J - \frac{2}{\sqrt{15}} G_3^J + \sqrt{\frac{10}{21}} G_5^J \end{aligned} \tag{84}$$

in the nonrelativistic limit. The covariant amplitudes are, using ω and τ's,

$$A_1(\lambda) = [\omega(\lambda) : \phi^*(\lambda) \cdot r]$$
$$A_3(\lambda) = [\cdot\,\omega(\lambda) \cdot \tau^{(3)}(0) : \phi^*(\lambda) \cdot]_W \, c_3 \, r^3 \tag{85}$$
$$A_5(\lambda) = [\omega(\lambda) : \tau^{(5)}(0) \therefore \phi^*(\lambda)] \, c_5 \, r^5$$

for $\ell = 1$, $\ell = 3$ and $\ell = 5$, respectively. The notation $[\cdot \quad \cdot]_W$ indicates that the first and the last free indices within $[\quad]$ are to be contracted with the modified metric $\tilde{g}(W)$. The symbol $\therefore$ stands for contraction over three neighboring indices. One finds

$$\begin{aligned} F_2^J &= \frac{1}{\sqrt{3}} \left(g_1 - \frac{2}{5} g_3 r^2 + \frac{2}{21} g_5 4^4 \right) r \\ F_1^J &= \sqrt{\frac{2}{15}} \, \gamma_s \left(2g_1 + \frac{1}{5} g_3 r^2 - \frac{10}{21} g_5 r^4 \right) r \\ F_0^J &= \sqrt{\frac{3}{5}} \left\{ g_1 \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) + \frac{4}{15} g_3 \left(\frac{3}{2}\gamma_s^2 - \frac{1}{2} \right) r^2 + \frac{20}{63} g_5 \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) r^4 \right\} r \end{aligned} \tag{86}$$

where g_1, g_3 and g_5 are arbitrary constants. If one takes the limit $\gamma_s \to 1$, then the following substitutions

$$G_1^J = \sqrt{\frac{7}{3}} \, g_1, \qquad G_3^J = -\frac{2}{5} \, g_3 \quad \text{and} \quad G_5^J = 2\sqrt{\frac{2}{63}} g_5 \tag{87}$$

transforms (86) into (84).

With the introduction of projection operators, one finds from (63)

$$\begin{aligned} \sqrt{2} F_2^J &= \left(\sqrt{\frac{2}{7}} g_1 + \sqrt{\frac{2}{3}} g_3 \, r^2 + \frac{1}{\sqrt{21}} g_5 \, r^4 \right) r \\ \sqrt{2} F_1^J &= \left(\frac{4}{\sqrt{35}} g_1 - \frac{1}{\sqrt{15}} g_3 \, r^2 - \sqrt{\frac{10}{21}} g_5 \, r^4 \right) r \, \gamma_s \\ F_0^J &= \left(\frac{3}{\sqrt{35}} g_1 - \frac{2}{\sqrt{15}} g_3 \, r^2 + \sqrt{\frac{10}{21}} g_5 \, r^4 \right) r \, \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) \end{aligned} \tag{88}$$

One is now confronted with a crucial difference between F_0^J in (86) and (88): it is seen that the functional forms for γ_s can be different for $\ell = 1$, 3 or 5 with the amplitudes constructed directly out of ω, whereas they are the same when the intermediate wave function χ is used instead.

IX. $H \to W^+W^-$

Consider the decay of a Higgs boson particle into two gauge bosons. The total intrinsic spin S can be 0, 1 or 2, while the orbital angular momentum ℓ can also be 0, 1 or 2. But since H is a scalar particle, one must have $\ell = S = 0$, $\ell = S = 1$ or $\ell = S = 2$. For the purpose of illustration, it is assumed that the Higgs boson decay can be parity nonconserving. In the nonrelativistic limit, the helicity-coupling amplitudes are given by

$$\begin{aligned} F^J_{\pm\pm} &= \sqrt{\frac{1}{3}} G^J_{00} \pm \frac{1}{\sqrt{2}} G^J_{11} + \sqrt{\frac{1}{6}} G^J_{22} \\ F^J_{00} &= -\sqrt{\frac{1}{3}} G^J_{00} + \sqrt{\frac{2}{3}} G^J_{22} \end{aligned} \tag{89}$$

where $J = 0$. Note that (3) is no longer valid, if G^J_{11} is nonzero, and that Bose symmetry for two gauge bosons is automatic in this formulation.

The decay amplitudes may be written

$$\begin{aligned} A_{00}(\lambda\nu) &= [\varepsilon(-\nu) \cdot \omega(\lambda)] \\ A_{11}(\lambda\nu) &= [p\ \varepsilon(-\nu)\ r\ \omega(\lambda)] \\ A_{22}(\lambda\nu) &= [\varepsilon(-\nu) \cdot \tau^{(2)}(0) \cdot \omega(\lambda)]\, c_2\, r^2 \end{aligned} \tag{90}$$

The amplitudes (90) lead to

$$\begin{aligned} F^J_{\pm\pm} &= -g_{00} \pm g_{11}\, W_H\, r + \frac{1}{3} g_{22}\, r^2 \\ F^J_{00} &= \gamma^2 \left(g_{00} + \frac{2}{3} g_{22}\, r^2 \right) \end{aligned} \tag{91}$$

where W_H is the Higgs boson mass and one has put $\gamma = \gamma_s = \gamma_\sigma$ in the H rest frame. Alternatively, one may use (63) to write

$$\begin{aligned} F^J_{\pm\pm} &= \sqrt{\frac{1}{3}} g_{00} \pm \frac{1}{\sqrt{2}} g_{11}\, W_H\, r + \sqrt{\frac{1}{6}} g_{22}\, r^2 \\ F^J_{00} &= \gamma^2 \left(-\sqrt{\frac{1}{3}} g_{00} + \sqrt{\frac{2}{3}} g_{22}\, r^2 \right) \end{aligned} \tag{92}$$

This is of course equivalent to (91) with a redefinition of the g's. In a phenomenological approach adopted here, it is seen that the Higgs boson decay into gauge bosons depends on three parameters g_{00}, g_{11} and g_{22}, which can depend in general on the Higgs boson and gauge-boson masses.

The decay probability is, summed over the helicities,

$$I \propto |F^J_{++}|^2 + |F^J_{00}|^2 + |F^J_{--}|^2 \tag{93}$$

since the D function is an identity for $J = 0$. If the Higgs boson mass W_H is very much larger than the W mass and the parameter g's depend weakly on the masses, then the decay probability I is dominated by $|F^J_{00}|^2$ only, i.e., both W^+ and W^- have zero helicities. Consider now the decay $H \to \gamma\gamma$. In this case I is given by $|F^J_{\pm\pm}|^2$, i.e., both of the γ's are restricted to $\pm$ helicities. Note clear separation of the decay amplitudes for these two cases.

In the Standard Model, the decay amplitude is given by, in the lowest-order tree diagram, the Lorentz metric $\bar{g}_{\alpha\beta}$ itself, which is contained in the amplitude A_{00} in (90). Once again, within the context of the projection operators used in this paper, the appropriate Lorentz metric is the modified one, i.e., $\tilde{g}_{\alpha\beta}(W_H)$. It should be pointed out that parity violation can occur only through the fermion loop in the decay of Higgs boson, and therefore it is expected to be relatively small. If parity is conserved in the decay, then one must set $g_{11} = 0$, and one has $F^J_{++} = F^J_{--}$ in this case.

X. $J/\psi \to a_2(1320)\rho$

In order to further illustrate the techniques, one treats here a case in which both s and σ have spins greater than zero. This decay involves $S = 1$, 2 and 3 with ℓ taking on the values 0, 2 or 4.

The invariant amplitudes may be written, noting that χ is a tensor of rank 3,

$$\begin{aligned}
A_{01}(\lambda\nu) &= e^{(1)}(\lambda\nu)\ \chi^{(1)}_{\alpha\beta;\rho}(\delta)\ \tilde{g}^{\beta\rho}(W)\ \tilde{g}^{\alpha\gamma}(W)\ \phi^*_\gamma(\delta) \\
A_{2S}(\lambda\nu) &= e^{S}(\lambda\nu)\ \left[\tau^{(2)}(0) : \chi^S(\delta) \cdot \phi^*(\delta)\right]\ c_2\, r^2 \\
A_{43}(\lambda\nu) &= e^{(3)}(\lambda\nu)\ \left[\chi^{(3)}(\delta) \therefore \tau^{(4)}(0) \cdot \phi^*(\delta)\right]\ c_4\, r^4
\end{aligned} \tag{94}$$

where S can be 1, 2 or 3 and the first subscript of A stands for ℓ. One sees that there are five distinct amplitudes in the problem. $e^S(\lambda\nu)$ is a function already defined in (44). It should be noted that the rank-3 tensor χ is symmetric and traceless in the first two indices but *not* with the third (note the semicolon indicating this distinction).

If one insists on bypassing the projection operators, the amplitudes may be written

$$\begin{aligned}
A_{01}(\lambda\nu) &= [\varepsilon(-\nu)\cdot\omega(\lambda)\cdot\phi^*(\delta)] \\
A_{21}(\lambda\nu) &= [\varepsilon(-\nu)\cdot\omega(\lambda)\cdot\tau^{(2)}(0)\cdot\phi^*(\delta)]\, c_2\, r^2 \\
A_{22}(\lambda\nu) &= [\varepsilon(-\nu)\ \omega(\lambda)\cdot\tau^{(2)}(0)\ \phi^*(\delta)]\, c_2\, r^2 \\
A_{23}(\lambda\nu) &= [\varepsilon(-\nu)\cdot\tau^{(2)}(0)\cdot\omega(\lambda)\cdot\phi^*(\delta)]\, c_2\, r^2 \\
A_{43}(\lambda\nu) &= [\varepsilon(-\nu)\,\omega(\lambda)\therefore\tau^{(4)}(0)\cdot\phi^*(\delta)]\, c_4\, r^4
\end{aligned} \tag{95}$$

where the square bracket in A_{22} implies a contraction over four free indices with the totally antisymmetric rank-4 tensor [see (24)]. These lead to F^J's with functional forms on γ_s which are dependent on ℓ in general. As the above amplitudes are not unique, one may conclude that the resulting γ_s dependence is not unique either.

In a phenomenological approach, therefore, it may be more practical to simply read off the form of the helicity-coupling amplitudes from (63):

$$\begin{aligned}
\sqrt{2}F^J_{21} &= \sqrt{\frac{2}{5}}g_{01} + \sqrt{\frac{1}{5}}g_{21}\,r^2 + \sqrt{\frac{1}{3}}g_{22}\,r^2 + \sqrt{\frac{4}{105}}g_{23}\,r^2 + \sqrt{\frac{1}{35}}g_{43}\,r^4 \\
\sqrt{2}F^J_{10} &= \left(-\sqrt{\frac{1}{5}}g_{01} - \sqrt{\frac{1}{10}}g_{21}\,r^2 + \sqrt{\frac{1}{6}}g_{22}\,r^2 + \sqrt{\frac{32}{105}}g_{23}\,r^2 + \sqrt{\frac{8}{35}}g_{43}\,r^4\right)\gamma_s\gamma_\sigma \\
\sqrt{2}F^J_{11} &= \left(\sqrt{\frac{1}{5}}g_{01} - \sqrt{\frac{2}{5}}g_{21}\,r^2 + \sqrt{\frac{6}{35}}g_{23}\,r^2 - \sqrt{\frac{8}{35}}g_{43}\,r^4\right)\gamma_s \\
\sqrt{2}F^J_{0,-1} &= \left(\sqrt{\frac{1}{15}}g_{01} + \sqrt{\frac{1}{30}}g_{21}\,r^2 - \sqrt{\frac{1}{2}}g_{22}\,r^2 + \sqrt{\frac{8}{35}}g_{23}\,r^2 + \sqrt{\frac{6}{35}}g_{43}\,r^4\right)\left(\frac{2}{3}\gamma_s^2+\frac{1}{3}\right) \\
F^J_{00} &= \left(-\sqrt{\frac{2}{15}}g_{01} + \sqrt{\frac{4}{15}}g_{21}\,r^2 + \sqrt{\frac{9}{35}}g_{23}\,r^2 - \sqrt{\frac{12}{35}}g_{43}\,r^4\right)\left(\frac{2}{3}\gamma_s^2+\frac{1}{3}\right)\gamma_\sigma
\end{aligned} \tag{96}$$

If σ is a photon, then the second and the fifth equations above are absent, and the angular distribution depends in general on three F^J's. It is seen that there are nevertheless five g's; one is in fact confronted with three independent coefficients of r^2 for three F^J's. In principle, with sufficient statistics on the parent state with a finite width, one may be able to discern different r^2 dependence for each F^J.

XI. CONCLUSIONS

In this paper a general formalism is developed for constructing covariant helicity-coupling amplitudes $F^J_{\lambda\nu}$ in an arbitrary two-body decay $J \to s + \sigma$ (the spins are used to designate the particles as well—see Table II). The decay amplitudes are given as expansions in the total intrinsic spin S and pure orbital angular momentum ℓ. For the purpose, one has introduced intermediate wave functions $\chi(m_s)$ and $\tau(m)$, which are the tensor analogues of the ket states $|Sm_s\rangle$ and $|\ell m\rangle$. By requiring that they have vanishing time components in the J rest frame, the covariant decay amplitudes reduce to those involving three-vectors only in the J rest frame. This is a general rule without exception: disregard time components of all the four-vectors p (parent momentum), r (decay relative momentum), ω (wave function for decay product s), ε (wave function for decay product σ), and ϕ^* (wave function for parent particle) in the problem; replace $\tilde{g}(W)$ by δ_{ij} with $i,j = 1,2,3$ wherever an inner product appears in the amplitudes; and replace the Lorentz scalar $(pabc)$ by its three-vector counterpart $W(\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c})$ (W is the mass of the parent particle). This rule applies even to an S-wave decay, e.g. for $b_1(1235) \to \omega\pi$ the amplitude is $(\boldsymbol{\omega} \cdot \boldsymbol{\phi}^*)$ in the J rest frame (see Section VI).

The helicity coupling amplitudes have been shown to depend in general on four variables: W, r (in this case, magnitude of the three-vector), γ_s (the Lorentz factor for the decay product s) and γ_σ (the same for the decay product σ), with the latter three evaluated in the J rest frame. Note that, if the states J, s and σ have finite widths, then the masses W, m and μ themselves are all continuous variables nad hence should be treated as variables in the problem. In addition, it has been shown that the dependence on the Lorentz factors is present only if the spins s and σ are greater than zero. Indeed, if $s = \sigma = 0$, then the covariant helicity-coupling amplitude F^J_0 is merely given by r^J, identical to the result of the nonrelativistic formalism. It should be emphasized that the Zemach formalism [9] is essentially nonrelativistic, since the γ factors have been completely ignored in his formulation.

Energy dependence of the helicity-coupling amplitudes is a necessary consequence of the fact that the decay products s and σ have finite momenta in the J rest frame. Specifically, if one of the decay product 'σ' is a spin-0 particle, the helicity-coupling amplitude F^J_λ is in general a polynomial of order $s - |\lambda|$ in γ_s, where λ is the helicity of s. The functional dependence on γ_s is simple indeed if $s = 1$: it is a monomial. Thus F^J_0 is simply proportional to γ_s. Likewise, if $s = 2$, then F^J_1 is shown to be proportional to γ_s, but F^J_0 depends on a functional form of γ_s, different in general for each ℓ (see Section VIII for an example).

One of the main objectives of this paper has been to point out that a more systematic and appropriate way is to employ the projection operators for both s and σ, when their spins are greater than zero. This formalism naturally leads to overall multiplicative factors on the γ dependence which can be easily calculated. In addition, the formalism gives the helicity-coupling amplitudes in terms of the ℓS-coupling amplitudes, closely resembling those familiar in the nonrelativistic limit. This formula, given in (63) and (64), is the main result of this paper. The functional forms for γ have been shown explicitly for spins up to 4, and the general formula for arbitrary integer spin is given in (52). One may note that this formula is derived in a straightforward way from the general expression for a wave function of arbitrary integer spin, given in (33). To the best knowledge of the author, such a general expression for the wave function has been worked out for the first time in this paper.

In the limit $W \to \infty$, the variables r and γ's also go to ∞. The factor r^ℓ is in practice always replaced by a Blatt-Weisskopf function (see [1]) which approaches a constant as $r \to \infty$. The Lorentz factors are kinematical in origin and therefore must remain undamped in the covariant amplitudes. One sees then that the angular distribution, in the limit $W \to \infty$, is determined by only one F^J_{ab} where $a = \min|\lambda|$ and $b = \min|\nu|$ allowed in the problem. For example, consider the decay of a Higgs boson particle into two gauge bosons. As the Higgs boson mass tends to infinity, the decay amplitude is essentially determined by a single helicity-coupling amplitude F^J_{00}; in other words, the gauge bosons behave as if they were scalar particles (Goldstone bosons) [10].

The decays involving a photon in the final state should be treated in the same way: the decay amplitudes are given an expansion in a state of definite S and ℓ, as if the photon were a massive vector particle, e.g. $\rho(770)$ or $\omega(782)$. One then imposes a condition that the photon wave function have no zero z component. The intermediate wave functions χ and τ, required for this procedure, have nothing to do with the photon; they correspond to those of unobserved (and massive) spin-1 or higher-spin particles, defined to have zero time components in the parent rest frame. This approach allows for photons to be treated in exactly the same way as massive particles—appropriate for helicity-coupling amplitudes.

The unknowns in the decay problem, denoted as g_{ij}'s in this paper throughout (or simply g_i's depending on the problem), have been treated as constants. It should be clear, however, that one has chosen here a *model*—one which satisfies Lorentz invariance, incorporating the concept of definite S and ℓ—but a *model* nonetheless. In general, the g_{ij}'s should be functions of invariant variables in the problem, but—in the abence of dynamics—the functions are unkown. It is shown that there exists a one-to-one correspondence between the number of the $F^J_{\lambda\nu}$'s and g_{ij}'s. This is certainly the case, without exceptions, for the decays involving massive particles. However, if one of the decay particles is a photon, then the number of the g_{ij}'s can exceed that of $F^J_{\lambda\nu}$'s, as shown in the last example in this paper.

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