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Covariant Helicity-coupling Amplitudes: A Practical Guide

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Abstract

This note is an updated version of a previous work by the author. Particular emphasis is given to providing key intermediate steps for many practical examples involving two-body decays.

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I. INTRODUCTION

The purpose of this note is to update a recent paper by the author [1] on 'helicity-coupling amplitudes in tensor formalism,' which dealt with the processes involving arbitrary two-body decays. It was shown in the paper that, if the daughter particle has spin 1 or higher, the helicity-coupling amplitudes depend in general on $\gamma = E/m$, where m is the mass of the daughter and E is its energy in the parent rest frame.

Many of the formulas contained in Ref. [1] are reproduced here for completeness. In addition, the same examples as in the original work are treated again, but with emphasis on providing key intermediate steps. Wave functions for spin-3 have been added in Section III; in addition, explicit forms for both spin-2 and spin-3 wave functions have been worked out. Section IV is new; one finds in it all the elementary Lorentz scalars needed for constructing covariant helicity-coupling amplitudes in arbitrary decay processes. Section XI is also new—a treatment is given of spin-3 states decaying into a spin-1 or spin-2 state and a spin-0 particle.

In this note one has systematically and exclusively utilized projection operators corresponding to pure intrinsic spin S and pure orbital angular momentum ℓ. This affords a general and efficient means of identifying each covariant amplitude in the problem. This method provides, in addition, a means of systematically handling all the decays which involve photons in the final state on an equal footing.

Finally, it should be noted that several new notations have been introduced in this note, in order to clarify the role of the modified Lorentz metric used in constructing covariant amplitudes.

II. HELICITY-COUPLING AMPLITUDES

Consider a state with spin(parity)=$J(\eta_J)$ decaying into two states with $s(\eta_s)$ and $\sigma(\eta_\sigma)$. The decay amplitudes are given, in the rest frame of J,

$$\begin{aligned}\mathcal{M}^J_{\lambda\nu}(\vartheta,\varphi;M) &\propto \langle \vartheta,\varphi,\lambda\nu|JM\lambda\nu\rangle\langle JM\lambda\nu|\ \mathcal{M}\ |JM\rangle \\ &\propto D^{J*}_{M\delta}(\varphi,\vartheta,0)F^J_{\lambda\nu}\end{aligned} \tag{1}$$

where $\mathcal{M}$ is the invariant operator for the decay, and λ and ν are the helicities of the two final-state particles 's' and 'σ' with $\delta = \lambda - \nu$. The M is the z-component of the spin J in a coordinate system fixed in the production system. The helicities λ and ν are rotational invariants by definition. For the decay amplitude $\mathcal{M}$ one has adopted a system of notation in which rotational invariants are given as indices while the non-invariants are listed as arguments. The direction of the break-up momentum of the decaying particle s is given by the angles ϑ and φ in the J rest frame. Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be the coordinate system fixed in the J rest frame. It is important to recognize, for applications to sequential decays, the exact nature of the body-fixed (helicity) coordinate system implied by the arguments of the D-function given above. Let $\hat{x}_h$, $\hat{y}_h$ and $\hat{z}_h$ be the helicity coordinate system fixed by the s decay. Then by definition $\hat{z}_h$ describes the direction of the s in the J rest frame and the y-axis is given by $\hat{y}_h \propto \hat{z} \times \hat{z}_h$ and $\hat{x}_h = \hat{y}_h \times \hat{z}_h$.

The helicity-coupling amplitude F^J given by

$$F^J_{\lambda\nu} \;\propto\; \langle JM\lambda\nu|\; \mathcal{M} \;|JM\rangle \tag{2}$$

is a rotational invariant. Parity conservation in the decay leads to the relationship

$$F^J_{\lambda\nu} = \eta_J \eta_s \eta_\sigma (-)^{J-s-\sigma} F^J_{-\lambda-\nu} \tag{3}$$

while, if the decay products 's' and 'σ' are identical, the following additional relationship holds

$$F^J_{\lambda\nu} = (-)^J F^J_{\nu\lambda} \tag{4}$$

for both integer and half-integer spins.

If the decay particles are both massive, the decay amplitude may be given in the ℓS-coupling scheme

$$\begin{aligned}\mathcal{M}^J_{\ell S}(\vartheta,\varphi;m_1m_2M) &\propto \sum_{mm_s} \langle\vartheta,\varphi,m_1m_2|\ell mSm_s\rangle\langle\ell mSm_s|\; \mathcal{M} \;|JM\rangle \\ &\propto G^J_{\ell S}\;(sm_1\,\sigma m_2|Sm_s)\sum_m\;(\ell m\,Sm_s|JM)\;Y^\ell_m(\vartheta,\varphi)\end{aligned} \tag{5}$$

where $G^J_{\ell S}$ is the ℓS-coupling amplitude given by

$$\langle\ell mSm_s|\; \mathcal{M} \;|JM\rangle \;=\; (\ell m\,Sm_s|JM)\;G^J_{\ell S} \tag{6}$$

It should be noted that m_1 and m_2 are measured in the canonical rest systems (reached by pure time-like Lorentz transformations from the J rest frame) of the two decay products. The $(\ell_1 m_1\,\ell_2 m_e|\ell_3 m_3)$ stands for the usual Clebsch-Gordan coefficients. The decay amplitude in the ℓS- coupling scheme contains two extra variables m_1 and m_2 that are not rotational invariants [compare with (1)].

It is instructive to write down (1) and (5) again for $\vartheta = \varphi = 0$

$$\begin{aligned}\mathcal{M}^J_{\lambda\nu}(0,0;\delta) &\propto F^J_{\lambda\nu} \\ \mathcal{M}^J_{\ell S}(0,0;m_1m_2m_s) &\propto G^J_{\ell S}\;(sm_1\,\sigma m_2|Sm_s)\;(\ell 0\,Sm_s|Jm_s)\end{aligned} \tag{7}$$

By setting $m_1 = \lambda$, $m_2 = -\nu$ and $m_s = \delta$ in the latter expression above, one finds that the helicity-coupling amplitudes F^J are related to the ℓS-coupling amplitudes $G^J_{\ell S}$ via

$$F^J_{\lambda\nu} = \sum_{\ell S}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}}(\ell 0\,S\delta|J\delta)(s\lambda\,\sigma{-}\nu|S\delta)\;G^J_{\ell S} \tag{8}$$

where the coupling amplitudes have been given the normalization

$$\sum_{\ell S} |G^J_{\ell S}|^2 = \sum_{\lambda\nu} |F^J_{\lambda\nu}|^2 \tag{9}$$

The formula (8) for the helicity-coupling amplitudes results from the usual scheme of coupling of the angular momenta but with the z-axis chosen along the break-up momentum of the decay product 's'. Note that the orbital angular momentum ℓ has the zero z-component in this case and the particle 'σ' has the z-component $-\nu$.

The decay amplitude (1) is simply given by the helicity-coupling amplitude itself if one sets $\vartheta = \varphi = 0$, as shown in (7). It is obvious now that the helicity-coupling amplitudes can be derived from the tensor formalism by restricting to the four-vectors defined along the z-axis. Let p, q and k be the four-momenta for the states 'J', 's' and 'σ' with masses W, m and μ

$$p^\alpha = (p_0, \boldsymbol{p}), \quad p^2 = W^2, \quad q^\alpha = (q_0, \boldsymbol{q}), \quad q^2 = m^2, \quad k^\alpha = (k_0, \boldsymbol{k}), \quad k^2 = \mu^2 \tag{10}$$

and let $r = q - k$ be the break-up four-momentum. Using the Lorentz metric $g_{\alpha\beta}$

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11}$$

one has

$$p_\alpha = g_{\alpha\beta} p^\beta = (p_0, -\boldsymbol{p}\ ) \tag{12}$$

and similarly for the other four-vectors. In this note, one uses the notations p, q, k and r to stand for <u>both</u> the four-momenta and the magnitudes of the 3-momenta. One can then define the following unitless quantities derived from them:

$$\gamma_s = \frac{q_0}{m}, \quad \gamma_s\beta_s = \frac{q}{m}, \quad \gamma_\sigma = \frac{k_0}{\mu} \quad \text{and} \quad \gamma_\sigma\beta_\sigma = \frac{k}{\mu} \tag{13}$$

One may now write an explicitly covariant expression (Lorentz scalar) for the helicity-coupling amplitudes

$$F^J_{\lambda\nu} = \sum_\alpha g_\alpha A_\alpha(\lambda\nu) \equiv \sum_\alpha g_\alpha A_\alpha\{p^n, r^\ell, \omega(\lambda), \varepsilon(-\nu), \phi^*(\delta)\} \tag{14}$$

as a sum of functions A_α of five variables, provided all the four-momenta are defined along the z-axis (i.e. no x- and y-components). As the momenta involved are all parallel with the z-axis, this formula merely gives the momentum dependence of the helicity-coupling amplitudes but no angular dependence, as this is already contained in the D-function in the expression (1). The variables α stand for the set $\{\ell, S\}$, and the constants g_α are the analogue of the $G^J_{\ell S}$ in (8).

The covariant function A_α depends on p and r as well as the momentum-space wave functions (or tensors) $\phi^*(\delta)$, $\omega(\lambda)$ and $\varepsilon(-\nu)$ for the particles 'J', 's' and 'σ', where δ, λ and $-\nu$ are the z-components of spin as defined before. Note that the complex conjugate of the J wave function appears in the above formula: it represents the initial state while those of 's' and 'σ' correspond to the final states. As shown with examples in later sections, one may set $n = 1$ or $n = 0$ without loss of generality, depending on the intrinsic parities involved. In another words, the four-vector p is used in the covariant amplitudes at most once, if necessary, in order to satisfy the requirement of parity conservation. The covariant function A_α can depend on any multiples (up to ℓ) of r, reflecting orbital angular momenta allowed in the decay.

The formula (14) implies that the energy dependence on F^J is complex, with the energies q_0 and k_0 appearing in the expression in addition to r, as the wave functions for 's' and 'σ' are evaluated in the J rest frame, not in their respective rest frames. One may also note that, in addition, the masses W, m and μ appear in the expression for F^J as a consequence of the covariant description of the decay amplitudes.

A summary of the notations used in this note is given in Table I.

Table I. Two-body decay: $J \to s + \sigma$

	Parent	Daughter 1	Daughter 2
Spin	J	s	σ
Parity	η_J	η_s	η_σ
Helicity		λ	ν
Momentum	p	q	k
Energy	p_0	q_0	k_0
Mass	W	m	μ
Energy/Mass		γ_s	γ_σ
Wave function	$\phi^*(\lambda-\nu)$	$\omega(\lambda)$	$\varepsilon(-\nu)$

III. POLARIZATION FOUR-VECTORS

The polarization four-vector appropriate for the particle J is given first in the following; those for the decay products 's' and 'σ' are of course similar.

The polarization four-vectors are in reality the spin-1 wave functions imbedded in the momentum space. Thus, under a general rotation R, the four-vectors transform according to

$$\phi^\alpha(m) \to \sum_{m'} \phi^\alpha(m') D^1_{m'm}(R) \tag{15}$$

where m and m' are the z-component of spin in an arbitrary coordinate system. In order for this to hold, the rest-frame wave functions should be given by

$$\begin{aligned} \phi^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}}\ (0;\ 1,\ \pm i,\ 0) \\ \phi^\alpha(0) &= \qquad (0;\ 0,\quad 0,\ 1) \end{aligned} \tag{16}$$

where the time component is defined to be zero in the rest frame. These polarization four-vectors satisfy

$$\begin{aligned} p_\alpha \phi^\alpha(m) &= \ 0 \\ \phi^*_\alpha(m)\phi^\alpha(m') &= -\delta_{mm'} \\ \sum_m \phi_\alpha(m)\phi^*_\beta(m) &= \ \tilde{g}_{\alpha\beta} \end{aligned} \tag{17}$$

where

$$\tilde{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta}(W) = -g_{\alpha\beta} + \frac{p_\alpha p_\beta}{W^2} \tag{18}$$

The last equation of (17) is the usual projection operator for spin-1 states. Note that in the J rest frame $\tilde{g}$ takes on the form

$$\tilde{g}_{\alpha\beta}(W) = \tilde{g}^{\alpha\beta}(W) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

The argument W will be dropped whenever there is no ambiguity; however, one must remember that there exist in the problem two additional $\tilde{g}$'s, i.e. $\tilde{g}(m)$ and $\tilde{g}(\mu)$ for the states s and σ, respectively. An important Lorentz scalar for the decay amplitudes involves contraction of any two four-vectors a and b using the modified Lorentz metric $\tilde{g}(W)$:

$$(a \cdot \tilde{g} \cdot b) = a^\alpha\, \tilde{g}_{\alpha\beta}(W)\, b^\beta = (\tilde{g} :\ ab) = (\tilde{a} \cdot b) = (a \cdot \tilde{b}) = -(\tilde{a} \cdot \tilde{b}) = (\boldsymbol{a} \cdot \boldsymbol{b}) \tag{20}$$

to be compared with

$$(a \cdot g \cdot b) = a^\alpha\, g_{\alpha\beta}\, b^\beta = (g :\ ab) = (a \cdot b) = a_0 b_0 - (\boldsymbol{a} \cdot \boldsymbol{b}) \tag{21}$$

where a_0, b_0, $\boldsymbol{a}$ and $\boldsymbol{b}$ are evaluated in the J rest frame. The colon stands for contraction of two rank-2 tensors. The modified four-vectors are

$$\begin{aligned} \tilde{a} = \tilde{g} \cdot a \quad \Rightarrow \quad \tilde{a}_\alpha &= \tilde{g}_{\alpha\beta}\, a^\beta = (0; \boldsymbol{a}) \\ \tilde{a}^\alpha &= (0; -\boldsymbol{a}) \end{aligned} \tag{22}$$

and

$$\tilde{b} = \tilde{g} \cdot b \quad \Rightarrow \quad \begin{aligned} \tilde{b}_\alpha &= \tilde{g}_{\alpha\beta}\, b^\beta = (0; \boldsymbol{b}) \\ \tilde{b}^\alpha &= (0; -\boldsymbol{b}) \end{aligned} \tag{23}$$

Note the following useful relationship:

$$\begin{aligned} \tilde{g} \cdot \tilde{g} &= -\tilde{g} \\ \tilde{g} \cdot \tilde{g} \cdot \tilde{g} &= +\tilde{g} \end{aligned} \tag{24}$$

One has adopted here the usual convention that a dot ' $\cdot$ ' signifies contraction between an upper and lower four-vector indices or vice versa.

The property (15) implies that the spin-2 wave functions should be written

$$\phi^{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1\, 1m_2 | 2m) \phi^\alpha(m_1) \phi^\beta(m_2) \tag{25}$$

or

$$\begin{aligned} \phi^{\alpha\beta}(+2) &= \phi^\alpha(+)\, \phi^\beta(+) \\ \phi^{\alpha\beta}(+1) &= \frac{1}{\sqrt{2}} \left[\phi^\alpha(+)\, \phi^\beta(0) + \phi^\alpha(0)\, \phi^\beta(+) \right] \\ \phi^{\alpha\beta}(0) &= \frac{1}{\sqrt{6}} \left[\phi^\alpha(+)\, \phi^\beta(-) + \phi^\alpha(-)\, \phi^\beta(+) \right] + \sqrt{\frac{2}{3}} \phi^\alpha(0)\, \phi^\beta(0) \end{aligned} \tag{26}$$

and they transform, under the rotation R,

$$\phi^{\alpha\beta}(m) \rightarrow \sum_{m'} \phi^{\alpha\beta}(m') D^2_{m'm}(R) \tag{27}$$

These spin-2 tensors satisfy

$$\begin{aligned} p^\alpha \phi_{\alpha\beta}(m) &= 0 \\ \phi_{\alpha\beta}(m) &= \phi_{\beta\alpha}(m) \\ g^{\alpha\beta} \phi_{\alpha\beta}(m) &= 0 \end{aligned} \tag{28}$$

which state that the spin-tensors are orthogonal to the momentum, symmetric and traceless. These tensors satisfy, in addition,

$$\phi^*_{\alpha\beta}(m) \phi^{\alpha\beta}(m') = \delta_{mm'} \tag{29}$$

$$\sum_m \phi_{\alpha\beta}(m) \phi^*_{\gamma\delta}(m) = \frac{1}{2}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} + \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}) - \frac{1}{3}\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta} \tag{30}$$

One may note that the latter is the spin-2 projection operator.

The spin-3 wave functions can be constructed in a similar manner, starting with spin-2 and spin-1 wave functions, as follows:

$$\phi^{\alpha\beta\gamma}(m) = \sum_{m_1 m_2} (2m_1\, 1m_2|3m)\phi^{\alpha\beta}(m_1)\phi^{\gamma}(m_2) \tag{31}$$

or

$$\begin{aligned}
\phi^{\alpha\beta\gamma}(+3) &= \phi^{\alpha}(+)\,\phi^{\beta}(+)\,\phi^{\gamma}(+)\\
\phi^{\alpha\beta\gamma}(+2) &= \frac{1}{\sqrt{3}}\Big[\phi^{\alpha}(+)\,\phi^{\beta}(+)\,\phi^{\gamma}(0) + \phi^{\alpha}(+)\,\phi^{\beta}(0)\,\phi^{\gamma}(+) + \phi^{\alpha}(0)\,\phi^{\beta}(+)\,\phi^{\gamma}(+)\Big]\\
\phi^{\alpha\beta\gamma}(+1) &= \frac{1}{\sqrt{15}}\Big[\phi^{\alpha}(+)\,\phi^{\beta}(+)\,\phi^{\gamma}(-) + \phi^{\alpha}(+)\,\phi^{\beta}(-)\,\phi^{\gamma}(+) + \phi^{\alpha}(-)\,\phi^{\beta}(+)\,\phi^{\gamma}(+)\Big]\\
&\quad + \frac{2}{\sqrt{15}}\Big[\phi^{\alpha}(+)\,\phi^{\beta}(0)\,\phi^{\gamma}(0) + \phi^{\alpha}(0)\,\phi^{\beta}(+)\,\phi^{\gamma}(0) + \phi^{\alpha}(0)\,\phi^{\beta}(0)\,\phi^{\gamma}(+)\Big] \qquad (32)\\
\phi^{\alpha\beta\gamma}(0) &= \frac{1}{\sqrt{10}}\Big[\phi^{\alpha}(0)\,\phi^{\beta}(+)\,\phi^{\gamma}(-) + \phi^{\alpha}(0)\,\phi^{\beta}(-)\,\phi^{\gamma}(+) + \phi^{\alpha}(-)\,\phi^{\beta}(+)\,\phi^{\gamma}(0)\\
&\quad + \phi^{\alpha}(+)\,\phi^{\beta}(-)\,\phi^{\gamma}(0) + \phi^{\alpha}(+)\,\phi^{\beta}(0)\,\phi^{\gamma}(-) + \phi^{\alpha}(-)\,\phi^{\beta}(0)\,\phi^{\gamma}(+)\Big]\\
&\quad + \sqrt{\frac{2}{5}}\phi^{\alpha}(0)\,\phi^{\beta}(0)\,\phi^{\gamma}(0)
\end{aligned}$$

Note that these are orthogonal to p and are symmetric under interchange of any pairs in $\{\alpha, \beta, \gamma\}$. They are also traceless, i.e. they vanish under contraction with g or $\tilde{g}(W)$ for any pairs in $\{\alpha, \beta, \gamma\}$. One may ascertain, in addition, that

$$\phi^{*}_{\alpha\beta\gamma}(m)\phi^{\alpha\beta\gamma}(m') = -\delta_{mm'} \tag{33}$$

$$\begin{aligned}
\sum_m \phi_{\alpha\beta\gamma}(m)\phi^{*}_{\delta\mu\nu}(m) &= \frac{1}{3}(\tilde{g}_{\alpha\delta}\tilde{g}_{\beta\mu}\tilde{g}_{\gamma\nu} + \tilde{g}_{\alpha\mu}\tilde{g}_{\beta\nu}\tilde{g}_{\gamma\delta} + \tilde{g}_{\alpha\nu}\tilde{g}_{\beta\delta}\tilde{g}_{\gamma\mu})\\
&\quad - \frac{1}{5}(\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta}\tilde{g}_{\mu\nu} + \tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\mu}\tilde{g}_{\nu\delta} + \tilde{g}_{\gamma\alpha}\tilde{g}_{\beta\nu}\tilde{g}_{\delta\mu}) \qquad (34)
\end{aligned}$$

The latter is the spin-3 projection operator.

In order to find connection of the tensor formalism with that of the ℓS-coupling scheme, one needs to develop the concept of total intrinsic spin S formed out of the 's' and 'σ' polarization four-vectors and that of the pure orbital angular momentum ℓ built out of r. The key to a solution is contained in (5). One sees that this formula involves projection operators

$$\begin{aligned}
P^{(S)} &= \sum_{m_s} |Sm_s\rangle\langle Sm_s| \qquad (35)\\
P^{(\ell)} &= \sum_{m} |\ell m\rangle\langle \ell m|
\end{aligned}$$

Consider now a wave function $\chi(m_s)$ which is to form the basis for constructing the ket state $|Sm_s\rangle$. One demands that this wave function have zero time component in the J rest frame (very similar to ϕ) and the space components be given as in (16). The goal is to form rank-2 tensors, when coupled to ω and ε, project out pure spin S.

The desired rank-2 wave functions for spin-0, spin-1 and spin-2 are

$$
\begin{aligned}
\chi^{(0)}_{\alpha\beta}(0) &= \sum_{m_1 m_2} (1m_1\, 1m_2|00)\chi_\alpha(m_1)\chi_\beta(m_2) \\
\chi^{(1)}_{\alpha\beta}(m) &= \sum_{m_1 m_2} (1m_1\, 1m_2|1m)\chi_\alpha(m_1)\chi_\beta(m_2) \\
\chi^{(2)}_{\alpha\beta}(m) &= \sum_{m_1 m_2} (1m_1\, 1m_2|2m)\chi_\alpha(m_1)\chi_\beta(m_2)
\end{aligned} \tag{36}
$$

as analogue of the ket states $|Sm_s\rangle$. It is seen that spin-1 tensors are anti-symmetric while the spin-2 tensors are symmetric and traceless. The projection operator [see (30) for spin-2] given by

$$
P^{(S)}_{\alpha\beta\gamma\delta} = \sum_m \chi^{(S)}_{\alpha\beta}(m)\chi^{(S)*}_{\gamma\delta}(m) \tag{37}
$$

leads to, after some algebra,

$$
\begin{aligned}
P^{(0)}_{\alpha\beta\gamma\delta} &= \frac{1}{3}\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta} \\
P^{(1)}_{\alpha\beta\gamma\delta} &= \frac{1}{2}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}) \\
P^{(2)}_{\alpha\beta\gamma\delta} &= \frac{1}{2}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} + \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}) - \frac{1}{3}\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta}
\end{aligned} \tag{38}
$$

These result in the following pure-spin tensors:

$$
\begin{aligned}
\psi^{(0)}_{\alpha\beta} &= (\tilde{\omega}\cdot\varepsilon)\tilde{g}_{\alpha\beta} \\
\psi^{(1)}_{\alpha\beta} &= \tilde{\omega}_\alpha\tilde{\varepsilon}_\beta - \tilde{\varepsilon}_\alpha\tilde{\omega}_\beta \\
\psi^{(2)}_{\alpha\beta} &= \tilde{\omega}_\alpha\tilde{\varepsilon}_\beta + \tilde{\varepsilon}_\alpha\tilde{\omega}_\beta - \frac{2}{3}(\tilde{\omega}\cdot\varepsilon)\tilde{g}_{\alpha\beta}
\end{aligned} \tag{39}
$$

where, as given in (20),

$$
(\tilde{\omega}\cdot\varepsilon) = (\omega\cdot\tilde{\varepsilon}) = (\omega\cdot\tilde{g}\cdot\varepsilon) \tag{40}
$$

and

$$
\begin{aligned}
\tilde{\omega} = \tilde{g}\cdot\omega \quad \Rightarrow \quad \tilde{\omega}_\alpha &= \tilde{g}_{\alpha\beta}\omega^\beta \\
&= -\omega_\alpha + \frac{(p\cdot\omega)}{W^2}p_\alpha
\end{aligned} \tag{41}
$$

and similarly for $\tilde{\varepsilon}$ for any z-components of spin for ω and ε. Note that there exist three independent space components in an anti-symmetric tensor, while there are five independent space components in a symmetric and traceless tensor—reflecting spin states of spin-1 and spin-2 wave functions.

The analogue of the ket state $|\ell m\rangle$ may be constructed out of yet another polarization four-vector $\tau(m)$, defined to have zero time component in the J rest frame, since the orbital angular momentum is defined only in this frame. A 'pure' S-wave is characterized simply by the absence of $\boldsymbol{r}$ in the amplitude in the J rest frame. The spin-1 projection operator corresponding to $\tau(m)$ has already been given in the last equation of (17); this implies that the Lorentz metric, if required, should be replaced by the modified form of (18). The break-up four-momentum for $\ell = 1$ is then given by

$$\tilde{r} = \tilde{g} \cdot r \quad \Rightarrow \quad \tilde{r}_\alpha = \tilde{g}_{\alpha\beta} r^\beta \qquad (42)$$
$$= -r_\alpha + \left(\frac{m^2 - \mu^2}{W^2} \right) p_\alpha$$

with <u>zero</u> time component in the J rest frame.

The spin-2 projection operator resulting from $\tau(m)$ has also been worked out—see (30). A pure D-wave orbital angular momentum is thus represented by a rank-2 tensor

$$\tilde{t}^{(2)}_{\alpha\beta} = \tilde{r}_\alpha \tilde{r}_\beta - \frac{1}{3}(r \cdot \tilde{r}) \tilde{g}_{\alpha\beta} \qquad (43)$$

where, as given in (20),

$$(r \cdot \tilde{r}) = (\tilde{r} \cdot r) = (r \cdot \tilde{g} \cdot r) \qquad (44)$$

Similar technique leads to a rank-3 tensor for a pure F-wave

$$\tilde{t}^{(3)}_{\alpha\beta\gamma} = \tilde{r}_\alpha \tilde{r}_\beta \tilde{r}_\gamma - \frac{1}{5}(r \cdot \tilde{r})(\tilde{g}_{\alpha\beta}\tilde{r}_\gamma + \tilde{g}_{\beta\gamma}\tilde{r}_\alpha + \tilde{g}_{\gamma\alpha}\tilde{r}_\beta) \qquad (45)$$

A rank-4 tensor for a pure G-wave is

$$\begin{aligned} \tilde{t}^{(4)}_{\alpha\beta\gamma\delta} &= \tilde{r}_\alpha \tilde{r}_\beta \tilde{r}_\gamma \tilde{r}_\delta \\ &\quad - \frac{1}{7}(r \cdot \tilde{r})(\tilde{g}_{\alpha\beta}\tilde{r}_\gamma\tilde{r}_\delta + \tilde{g}_{\beta\gamma}\tilde{r}_\alpha\tilde{r}_\delta + \tilde{g}_{\gamma\alpha}\tilde{r}_\beta\tilde{r}_\delta + \tilde{g}_{\alpha\delta}\tilde{r}_\beta\tilde{r}_\gamma + \tilde{g}_{\beta\delta}\tilde{r}_\gamma\tilde{r}_\alpha + \tilde{g}_{\gamma\delta}\tilde{r}_\alpha\tilde{r}_\beta) \\ &\quad + \frac{1}{35}(r \cdot \tilde{r})^2(\tilde{g}_{\alpha\beta}\tilde{g}_{\gamma\delta} + \tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\delta} + \tilde{g}_{\gamma\alpha}\tilde{g}_{\beta\delta}) \end{aligned} \qquad (46)$$

It is obvious that use of these tensors would lead to differences in the helicity-coupling amplitudes when compared to those resulting from the simplest covariant amplitudes. They are pointed out with a number of examples in the following sections.

Both the spin-1 and spin-2 wave functions are generic, in the sense that they describe both positive and negative intrinsic-parity states. One merely needs to know how these wave functions transform under parity operation. For the purpose, the following notations and conventions are useful: Under parity operation, the momenta transform

$$p^\alpha \to \bar{p}^{\ \alpha} = (p_0, -\boldsymbol{p}\ ), \qquad \bar{p}_\alpha = g_{\alpha\beta}\bar{p}^{\ \beta} = (p_0, \boldsymbol{p}\ )$$

Let a, b, c and d be any four-vector. Using the short-hand notations,

$$(a \cdot b) = [a \cdot b] = g_{\alpha\beta} a^\alpha b^\beta, \qquad (abcd) = [abcd] = \epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^{\ \delta}$$

one finds

$$(a \cdot b) = (\bar{a} \cdot \bar{b}\ ), \qquad (abcd) = -(\bar{a}\bar{b}\bar{c}\bar{d}\ )$$

Define e via

$$e_\alpha = \epsilon_{\alpha\beta\gamma\delta} b^\beta c^\gamma d^{\ \delta}$$

which transform under parity operation as a pseudovector, i.e.

$$e^\alpha \to -\bar{e}^{\ \alpha} = (-e_0, \mathbf{e}\ )$$

As the spin-1 wave functions should be pseudovectors, they transform under parity in exactly the same way as the four-vector e, i.e.

$$\phi^\alpha(m) \to -\bar{\phi}^{\ \alpha}(m) \tag{47}$$

In another words, the time component changes sign, while the space components remain the same. Note the following transformation properties of the Lorentz scalars under parity operation:

$$(a \cdot \phi) \to -(\bar{a} \cdot \bar{\phi}) = -(a \cdot \phi), \qquad (e \cdot \phi) \to +(\bar{e} \cdot \bar{\phi}) = +(e \cdot \phi)$$

where a and e are any vector and pseudovectors. From (47), one sees that the spin-2 wave functions transform, under parity,

$$\phi^{\alpha\beta} \to \bar{\phi}^{\ \alpha\beta} \tag{48}$$

Suppose now that the second decay product 'σ' is a photon, i.e. $\mu = 0$. One may take for the photon polarization vector $\varepsilon(\pm)$ the four-vectors (16) but without the zero z-component of spin [i.e. $\varepsilon(0)$ not allowed], again setting the time component to zero (Coulomb gauge). Gauge invariance requires that any Lorentz scalar one writes down should vanish with the replacement $\varepsilon \to k$. The following two four-vectors, derived from the ε, satisfy this requirement:

$$\varepsilon^{(-)}_\alpha(\pm) = \epsilon_{\alpha\beta\gamma\delta} q^\beta k^\gamma \varepsilon^\delta(\pm) \tag{49}$$

$$\varepsilon^{(+)}_\alpha(\pm) = \varepsilon_\alpha(\pm) - \frac{[q \cdot \varepsilon(\pm)]}{(q \cdot k)} k_\alpha \tag{50}$$

Here the superscripts $(-)$ and $(+)$ denote a vector and a pseudovector, respectively. They transform, under parity,

$$\varepsilon^{(-)}_\alpha(\pm) \to +\bar{\varepsilon}^{(-)}_\alpha(\pm) \tag{51}$$

$$\varepsilon^{(+)}_\alpha(\pm) \to -\bar{\varepsilon}^{(+)}_\alpha(\pm) \tag{52}$$

IV. AMPLITUDES IN THE J REST FRAME

Let p, q and k stand for the four-momenta for the particles 'J', 's' and 'σ', and let ϕ, ω and ε stand for their spin-1 polarization four-vectors. In order to calculate the helicity-coupling amplitudes F^J in the prescription given in the expression (14), it is necessary to write down all the relevant momenta and the spin-1 polarization four-vectors along the z-axis, in the rest frame of the decaying particle:

$$\begin{array}{rcl} p^\alpha &=& (\quad W;\ 0,\ 0,\ \ 0\ ) \\ q^\alpha &=& (\quad q_0;\ 0,\ 0,\ \ q\ ) \\ k^\alpha &=& (\quad k_0;\ 0,\ 0,\ -q\ ) \\ r^\alpha &=& (\ q_0 - k_0;\ 0,\ 0,\ 2q\ ) \end{array} \tag{53}$$

where $W = q_0 + k_0$, $q_0 = \sqrt{m^2 + q^2}$, $k_0 = \sqrt{\mu^2 + q^2}$ and $r = q - k$.

The relevant polarization four-vectors are given by

$$\begin{array}{rcll} \phi^\alpha(\pm) &=& \mp\frac{1}{\sqrt{2}} & (\quad 0;\ 1,\ \pm i,\ \ 0\ ) \\ \phi^\alpha(0) &=& & (\quad 0;\ 0,\ \ 0,\ \ 1\ ) \\ \omega^\alpha(\pm) &=& \mp\frac{1}{\sqrt{2}} & (\quad 0;\ 1,\ \pm i,\ \ 0\ ) \\ \omega^\alpha(0) &=& & (\quad \gamma_s\beta_s;\ 0,\ \ 0,\ \gamma_s\ ) \\ \varepsilon^\alpha(\pm) &=& \mp\frac{1}{\sqrt{2}} & (\quad 0;\ 1,\ \pm i,\ \ 0\ ) \\ \varepsilon^\alpha(0) &=& & (-\gamma_\sigma\beta_\sigma;\ 0,\ \ 0,\ \gamma_\sigma\ ) \end{array} \tag{54}$$

Note that

$$[p \cdot \phi(\lambda)] = [q \cdot \omega(\lambda)] = [k \cdot \varepsilon(\lambda)] = 0$$

for any λ.

In the following sections, a number of examples involving photons will be considered. Let 'σ' be the photon. It is instructive to note here a general result concerning the Coulomb gauge adopted in this paper. Gauge invariance requires that one needs in the covariant amplitudes the photon polarization four-vectors given in (49) and (50). One may start out with the covariant amplitudes involving massive spin-1 states; it is seen that one must then perform the replacement [see (50) and (52)]

$$\varepsilon_\alpha(\pm) \to \varepsilon_\alpha(\pm) - \frac{[q \cdot \varepsilon(\pm)]}{(q \cdot k)} k_\alpha \tag{55}$$

in order to preserve parity conservation. In the particular case in which all the four-vectors are defined along the z-axis, one finds that the correction term of (55) is zero since $[q \cdot \varepsilon(\pm)] = 0$. It is instructive to work out the form of the vector $\varepsilon^{(-)}(\pm)$ in the J rest frame with the four-vectors as given in (54). One finds, after a straightforward algebra,

$$\varepsilon^{(-)}(\pm) = \pm iqW\varepsilon(\pm) \tag{56}$$

where q is the z-component of q^α given in (53). Note that the vector character of $\varepsilon^{(-)}(\pm)$ comes from the factor q above.

One therefore concludes that helicity-coupling amplitudes with photons can be treated in the same way as those with massive spin-1 particles in the final state—one merely uses the wave functions $\varepsilon(\pm)$ or with an additional factor $\pm iWr$ as required by parity conservation. This is equivalent to adding to the problem one unit of orbital angular momentum $\ell = 1$.

Consider an example of a decay for which $J = s = 1$, $\sigma = 0$ and $\ell = 0$ (S-wave). There are two ways of writing down this decay:

$$\begin{aligned} (\omega \cdot \phi^*) &= -(\boldsymbol{\omega} \cdot \boldsymbol{\phi}^*) \\ (\tilde{\omega} \cdot \phi^*) &= +(\boldsymbol{\omega} \cdot \boldsymbol{\phi}^*) \end{aligned} \tag{57}$$

where ϕ is the wave function for the parent particle J and ω is that for s. The 3-vectors on the right-hand side are evaluated in the J rest frame. The two equations above are the same except for the sign. Observe now the form of a pure D-wave orbital angular momentum (43): the second term with $\tilde{g}$ is the S-wave component contained in the rank-2 tensor $\tilde{r}\tilde{r}$. This shows that the second equation of (57) is in fact the proper one to use.

One is now ready to work out all the Lorentz scalars which occur in the calculation involving ω, ε, r and ϕ^*. A collection of bilinear products is given first. They are

$$\begin{aligned} a_1(m) &= [\tilde{r} \cdot \omega(m)] = [\boldsymbol{r} \cdot \boldsymbol{\omega}(m)] \\ a_1(\pm) &= 0 \\ a_1(0) &= \gamma_s\, r \end{aligned} \tag{58}$$

$$\begin{aligned} a_2(m) &= [\tilde{r} \cdot \varepsilon(m)] = [\boldsymbol{r} \cdot \boldsymbol{\varepsilon}(m)] \\ a_2(\pm) &= 0 \\ a_2(0) &= \gamma_\sigma\, r \end{aligned} \tag{59}$$

$$\begin{aligned} a_3(m) &= [\tilde{r} \cdot \phi^*(m)] = [\boldsymbol{r} \cdot \boldsymbol{\phi}^*(m)] \\ a_3(\pm) &= 0 \\ a_3(0) &= r \end{aligned} \tag{60}$$

and

$$\begin{aligned} b_1(m, m') &= [\tilde{\omega}(m) \cdot \varepsilon(m')] = [\boldsymbol{\omega}(m) \cdot \boldsymbol{\varepsilon}(m')] \\ b_1(m, m') &= 0, \quad \text{if} \quad m + m' \neq 0 \\ b_1(\pm, \mp) &= -1 \\ b_1(0, 0) &= \gamma_s\gamma_\sigma \end{aligned} \tag{61}$$

$$
\begin{aligned}
b_2(m,m') &= [\tilde{\omega}(m)\cdot\phi^*(m')] = [\boldsymbol{\omega}(m)\cdot\boldsymbol{\phi}^*(m')] \\
b_2(m,m') &= 0 \quad \text{for} \quad m \neq m' \\
b_2(\pm,\pm) &= +1 \\
b_2(0,0) &= \gamma_s
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
b_3(m,m') &= [\tilde{\varepsilon}(m)\cdot\phi^*(m')] = [\boldsymbol{\varepsilon}(m)\cdot\boldsymbol{\phi}^*(m')] \\
b_3(m,m') &= 0 \quad \text{for} \quad m \neq m' \\
b_3(\pm,\pm) &= +1 \\
b_3(0,0) &= \gamma_\sigma
\end{aligned} \tag{63}
$$

There are, in addition, Lorentz scalars involving the totally antisymmetric tensor of rank-4. Using the shorthand notation

$$
(\boldsymbol{abc}) = [\boldsymbol{abc}] = (\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}) \tag{64}
$$

$$
\begin{aligned}
c_1(m,m') &= [p\,\omega(m)\,r\,\phi^*(m')] = W[\boldsymbol{\omega}(m)\,\boldsymbol{r}\,\boldsymbol{\phi}^*(m')] \\
c_1(m,m') &= 0 \quad \text{for} \quad m \neq m' \\
c_1(\pm,\pm) &= \pm iWr \\
c_1(0,0) &= 0
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
c_2(m,m') &= [p\,\varepsilon(m)\,r\,\phi^*(m')] = W[\boldsymbol{\varepsilon}(m)\,\boldsymbol{r}\,\boldsymbol{\phi}^*(m')] \\
c_2(m,m') &= 0 \quad \text{for} \quad m \neq m' \\
c_2(\pm,\pm) &= \pm iWr \\
c_2(0,0) &= 0
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
c_3(m,m') &= [p\,\omega(m)\,r\,\varepsilon(m')] = W[\boldsymbol{\omega}(m)\,\boldsymbol{r}\,\boldsymbol{\varepsilon}(m')] \\
c_3(m,m') &= 0, \quad \text{if} \quad m+m' \neq 0 \\
c_3(\pm,\mp) &= \mp iWr \\
c_3(0,0) &= 0
\end{aligned} \tag{67}
$$

Note that $c_2(\pm,\pm)$ is proportional to $[\varepsilon^{(-)}(\pm)\cdot\phi^*(\pm)]$ and $c_3(\pm,\mp)$ to $[\omega(\pm)\cdot\varepsilon^{(-)}(\mp)]$ [see (56)].

$$\begin{aligned} e_1(m,m',m'') =& \; [p\,\omega(m)\,\varepsilon(m')\,\phi^*(m'')] = W[\boldsymbol{\omega}(m)\,\boldsymbol{\varepsilon}(m')\,\boldsymbol{\phi}^*(m'')] \\ e_1(m,m',m'') =& \; 0, \quad \text{if} \quad m+m' \neq m'' \\ e_1(\pm,\mp,0) =& \; \pm iW \\ e_1(\pm,0,\pm) =& \; \pm iW\gamma_\sigma \\ e_1(0,\pm,\pm) =& \; \mp iW\gamma_s \end{aligned} \tag{68}$$

When J, s or σ are greater 1, then the following normalizations are also needed:

$$\begin{aligned} f_1(m,m') &= [\tilde{\omega}(m)\cdot\omega(m')] = [\boldsymbol{\omega}(m)\cdot\boldsymbol{\omega}(m')] \\ f_1(m,m') &= 0, \quad \text{if} \quad m+m' \neq 0 \\ f_1(\pm,\mp) &= -1 \\ f_1(0,0) &= \gamma_s^2 \end{aligned} \tag{69}$$

$$\begin{aligned} f_2(m,m') &= [\tilde{\varepsilon}(m)\cdot\varepsilon(m')] = [\boldsymbol{\varepsilon}(m)\cdot\boldsymbol{\varepsilon}(m')] \\ f_2(m,m') &= 0, \quad \text{if} \quad m+m' \neq 0 \\ f_2(\pm,\mp) &= -1 \\ f_2(0,0) &= \gamma_\sigma^2 \end{aligned} \tag{70}$$

$$\begin{aligned} f_3(m,m') &= [\tilde{\phi}^*(m)\cdot\phi^*(m')] = [\boldsymbol{\phi}^*(m)\cdot\boldsymbol{\phi}^*(m')] \\ f_3(m,m') &= 0, \quad \text{if} \quad m+m' \neq 0 \\ f_3(\pm,\mp) &= -1 \\ f_3(0,0) &= 1 \end{aligned} \tag{71}$$

From these one finds the modified traces for spin-2 wave functions:

$$\begin{aligned} h_1(m) &= (\cdot\;\omega\;\cdot)_W = [\tilde{g}(W):\omega] = \sum_{m_1m_2}(1m_1\,1m_2|2m)\,[\tilde{\omega}(m_1)\cdot\omega(m_2)] \\ h_1(\pm) &= 0 \\ h_1(0) &= \sqrt{\frac{2}{3}}(\gamma_s^2-1) \end{aligned} \tag{72}$$

$$\begin{aligned} h_2(m) &= (\cdot\;\varepsilon\;\cdot)_W = [\tilde{g}(W):\varepsilon] = \sum_{m_1m_2}(1m_1\,1m_2|2m)\,[\tilde{\varepsilon}(m_1)\cdot\varepsilon(m_2)] \\ h_2(\pm) &= 0 \\ h_2(0) &= \sqrt{\frac{2}{3}}(\gamma_\sigma^2-1) \end{aligned} \tag{73}$$

$$
\begin{aligned}
h_3(m) &= (\cdot\ \phi^*\cdot)_W = [\tilde{g}(W):\phi^*] = \sum_{m_1 m_2} (1m_1\,1m_2|2m)\,[\tilde{\phi}^*(m_1)\cdot\phi^*(m_2)] \qquad (74)\\
h_3(\pm) &= 0\\
h_3(0) &= 0
\end{aligned}
$$

A shorthand notation $(\cdot \quad \cdot)_W$ is used to indicate contraction of the first and last indices of a tensor of rank-2 with the modified Lorentz metric $\tilde{g}(W)$. The colon stands for contraction of two rank-2 tensors. One notes that h_1 and h_2 go to zeros as γ's approach 1 (non-relativistic limit). It should be noted that $(g:\omega) = (g:\varepsilon) = 0$ with the usual Lorentz metric g but <u>not</u> with the modified metric $\tilde{g}(W)$.

The formulas (57)—(74) show that the amplitudes are to be constructed exclusively out of the 3-vectors defined in the J rest frame. It is seen that, in writing down covariant amplitudes in this scheme, one needs care in the use of $\tilde{g}(W)$: one should use only the Lorentz scalars $(a\cdot\tilde{g}\cdot b)$ and $(p\,a\,b\,c)$ with no 'tildas' in a, b or c.

V. SPIN-1 $\rightarrow$ SPIN-1 + SPIN-0

One starts out with simple but practical examples. Depending on the intrinsic parities involved, the treatment can be divided into two categories.

A. $\omega(1600) \rightarrow \rho + \pi$

Let 'J', 's' and 'σ' stand for the $\omega(1600)$, the ρ and the π. The net intrinsic parity is given by $\eta_J\eta_s\eta_\sigma = -1$. Due to parity conservation in the decay, the helicity-coupling amplitudes are as follows: $F_0^J = 0$ and $F_+^J = -F_-^J$. There is only one allowed orbital angular momentum, i.e. $\ell = 1$. The covariant amplitude takes on the form

$$A(\lambda) = [p\ \omega(\lambda)\ r\ \phi^*(\lambda)] \qquad (75)$$

where the vectors are defined along the z-axis. Evaluating $F_\lambda^J = gA(\lambda)$ in the J rest frame, one finds, from (65),

$$F_+^J = -F_-^J = gWr \qquad (76)$$

where $J = 1$ and the imaginary part (which always accompanies W) has been absorbed into g an arbitrary complex constant. The presence of r reflects the P-wave decay; the factor W results from the covariant description of the decay amplitude. Note the treatment given here applies equally well to the decay $\omega \rightarrow \pi^0 + \gamma$.

B. $b_1(1235) \to \omega + \pi$

Let 'J', 's' and 'σ' stand for the $b_1(1235)$, the ω and the π. The net intrinsic parity is given by $\eta_J\eta_s\eta_\sigma = +1$ and $F^J_\lambda = +F^J_{-\lambda}$. There are two allowed orbital angular momenta, i.e. $\ell = 0$ or $\ell = 2$. The helicity-coupling amplitudes have the following expansion:

$$\begin{aligned}\sqrt{2}F^J_+ &= \sqrt{\frac{2}{3}}G^J_0 + \sqrt{\frac{1}{3}}G^J_2 \\ F^J_0 &= \sqrt{\frac{1}{3}}G^J_0 - \sqrt{\frac{2}{3}}G^J_2\end{aligned} \tag{77}$$

where $J = 1$. Assume now that the $b_1(1235)$ decays predominantly $\omega\pi$. Then the $\omega\pi$ elastic scattering in the $J^P = 1^+$-wave is completely dominated by the $b_1(1235)$. It can be shown under this circumstances that the F^J_λ's are relatively real [see Section 5.2, Chung [2]], and hence the G^J_ℓ's are relatively real as well. According to the Particle Data Group [3], one has, experimentally,

$$\left|\frac{G^J_2}{G^J_0}\right| = 0.26 \pm 0.04 \tag{78}$$

There are two covariant decay amplitudes corresponding to S- and D-waves in the problem,

$$\begin{aligned}A_0 &= (\tilde{\omega} \cdot \phi^*) \\ A_2 &= (\omega \cdot \tilde{t}^{(2)} \cdot \phi^*)\end{aligned} \tag{79}$$

which are transformed into

$$\begin{aligned}A_0(\lambda) &= [\boldsymbol{\omega}(\lambda) \cdot \boldsymbol{\phi}^*(\lambda)] \\ A_2(\lambda) &= [\boldsymbol{r} \cdot \boldsymbol{\omega}(\lambda)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(\lambda)] - \frac{1}{3}r^2[\boldsymbol{\omega}(\lambda) \cdot \boldsymbol{\phi}^*(\lambda)]\end{aligned} \tag{80}$$

evaluated in the J rest frame. The helicity-coupling amplitudes are given by

$$F^J_\lambda = g_0 A_0(\lambda) + g_2 A_2(\lambda) \tag{81}$$

where g_0 and g_2 are arbitrary constants. Evaluating F's in the J rest frame, one obtains, from (58), (60) and (62),

$$\begin{aligned}F^J_+ &= g_0 - \frac{1}{3}g_2 r^2 \\ F^J_0 &= \gamma_s\left(g_0 + \frac{2}{3}g_2 r^2\right)\end{aligned} \tag{82}$$

where $J = 1$. Only in the limit $\gamma_s \to 1$, the expressions of (82) reduce to those of (77) with the replacement

$$G_0^J = \sqrt{3}\, g_0, \qquad G_2^J = -\sqrt{\frac{2}{3}}\, g_2\, r^2 \tag{83}$$

It is instructive to work out the angular distribution for this decay. Suppose now that ω decays into 3π and the orientation of its normal is given by (ϑ', φ') in the helicity coordinate system $\{\hat{x}_h, \hat{y}_h, \hat{z}_h\}$ as defined in Section II. Then the overall decay amplitude is

$$A^J(\vartheta, \varphi, \vartheta', \varphi', M) \propto \sum_\lambda\ D^{J*}_{M\lambda}(\varphi, \vartheta, 0) F^J_\lambda D^{s*}_{\lambda 0}(\varphi', \vartheta', 0) \tag{84}$$

In terms of the density matrix ρ defined in the J rest frame, the angular distribution I is

$$\begin{aligned} I(\vartheta, \varphi, \vartheta', \varphi') \propto \sum_{\substack{MM' \\ \lambda\lambda'}}\ & \rho_{MM'} D^{J*}_{M\lambda}(\varphi, \vartheta, 0) D^J_{M'\lambda'}(\varphi, \vartheta, 0) \\ & F^J_\lambda F^{J*}_{\lambda'} D^{s*}_{\lambda 0}(\varphi', \vartheta', 0) D^s_{\lambda' 0}(\varphi', \vartheta', 0) \end{aligned} \tag{85}$$

For the purpose of illustration, it is sufficient to take the special case in which $\rho_{00} = 1$ and all the other elements are zero. Then, after integrating over φ and φ', one finds

$$I(\vartheta, \vartheta') \propto \sum_\lambda\ [d^J_{0\lambda}(\vartheta)]^2\ |F^J_\lambda|^2\ [d^s_{\lambda 0}(\vartheta')]^2 \tag{86}$$

This leads to two very similar distributions

$$I(\vartheta) \propto |F^J_0|^2 \cos^2(\vartheta) + |F^J_+|^2 \sin^2(\vartheta) \tag{87}$$

$$I(\vartheta') \propto |F^J_0|^2 \cos^2(\vartheta') + |F^J_+|^2 \sin^2(\vartheta') \tag{88}$$

If one assumes that g_i's are relatively real, one obtains, from (82),

$$\begin{aligned} I(\vartheta) \propto\ & g_0^2[(\gamma_s^2 - 1)\cos^2(\vartheta) + 1] \\ & + \frac{2}{3} g_0 g_2\, r^2 [(2\gamma_s^2 + 1)\cos^2(\vartheta) - 1] \\ & + \frac{1}{9} g_2^2\, r^4 [(4\gamma_s^2 - 1)\cos^2(\vartheta) + 1] \end{aligned} \tag{89}$$

$$\begin{aligned} I(\vartheta') \propto\ & g_0^2[(\gamma_s^2 - 1)\cos^2(\vartheta') + 1] \\ & + \frac{2}{3} g_0 g_2\, r^2 [(2\gamma_s^2 + 1)\cos^2(\vartheta') - 1] \\ & + \frac{1}{9} g_2^2\, r^4 [(4\gamma_s^2 - 1)\cos^2(\vartheta') + 1] \end{aligned} \tag{90}$$

Two noteworthy results of this exercise are: (a)the S-wave term containing the factor g_0^2 is no longer isotropic in the cosines for $\gamma_s > 1$, and (b)the D-wave term with g_2^2 is a polynomial order 2 in the cosines, reflecting the fact that the parent particle has $J = 1$. It is important to work out the angular distribution again within the non-relativistic formalism of canonical quantization. Integrating over the variables corresponding to the ω decay and over φ, one finds, for $\rho_{00} = 1$,

$$I(\vartheta) \propto \sum_{\ell\ell'}\ G^J_\ell G^{J*}_{\ell'} \sum_m\ (\ell m\ s - m|J0)(\ell' m\ s - m|J0)\ Y^m_\ell(\vartheta, 0) Y^m_{\ell'}(\vartheta, 0) \tag{91}$$

With the substitutions (83), it can shown that this angular distribution reduces to that of (89) in the limit $\gamma_s \to 1$.

VI. SPIN-1 → SPIN-1 + SPIN-1

Four important and non-trivial examples are treated here; two of them are those of radiative J/ψ decays.

A. $J/\psi \to f_1(1420) + \omega$

Let 'J', 's' and 'σ' stand for the J/ψ, the $f_1(1420)$ and the ω. This decay, $1^- \to 1^+ + 1^-$, involves three helicity-coupling amplitudes F^J_{++}, F^J_{0+}, F^J_{+0}. There are two orbital angular momenta in this problem, i.e. $\ell = 0$ and $\ell = 2$. From (8), the F^J's are related to three $G^J_{\ell S}$'s through

$$\begin{aligned}
\sqrt{2}F^{(1)}_{++} &= \frac{1}{\sqrt{3}}G^{(1)}_{01} - \sqrt{\frac{2}{3}}\,G^{(1)}_{21} \\
\sqrt{2}F^{(1)}_{0+} &= \frac{1}{\sqrt{3}}G^{(1)}_{01} + \frac{1}{\sqrt{6}}G^{(1)}_{21} - \frac{1}{\sqrt{2}}G^{(1)}_{22} \\
\sqrt{2}F^{(1)}_{+0} &= \frac{1}{\sqrt{3}}G^{(1)}_{01} + \frac{1}{\sqrt{6}}G^{(1)}_{21} + \frac{1}{\sqrt{2}}G^{(1)}_{22}
\end{aligned} \tag{92}$$

Note that these satisfy (9).

The amplitude corresponding to $\ell = 0$ and $S = 1$ can be constructed out of $\psi^{(1)}$ [see (39)], p and ϕ^* using the totally anti-symmetric tensor. Similarly, the amplitude for $\ell = 2$ and $S = 1$ is built up out of p, $\tilde{t}^{(2)}$, $\psi^{(1)}$ and ϕ^*, while that for $\ell = 2$ and $S = 2$ may be made out of p, $\tilde{t}^{(2)}$, $\psi^{(2)}$ and ϕ^*. One finds

$$\begin{aligned}
A_1 &= \frac{1}{2}(p\,\psi^{(1)}\,\phi^*) \\
A_2 &= -(p\,\tilde{t}^{(2)}\cdot\psi^{(1)}\,\phi^*) \\
A_3 &= (p\,\tilde{t}^{(2)}\cdot\psi^{(2)}\,\phi^*)
\end{aligned} \tag{93}$$

Note that one could have written the second amplitude above in a different way:

$$A'_2 = (p\,\psi^{(1)}\,\tilde{t}^{(2)}\cdot\phi^*) \tag{94}$$

It is shown in the appendix of ref. [1] that A'_2 is proportional to A_2. The covariant amplitudes may now be expressed as

$$\begin{aligned}
A_1(\lambda\nu) &= [p\,\omega(\lambda)\varepsilon(-\nu)\phi^*(\lambda-\nu)] \\
A_2(\lambda\nu) &= [p\,\omega(\lambda)\,r\,\phi^*(\lambda-\nu)][\tilde{r}\cdot\varepsilon(-\nu)] - [p\,\varepsilon(-\nu)\,r\,\phi^*(\lambda-\nu)][\tilde{r}\cdot\omega(\lambda)] \\
&\quad - \frac{2}{3}(r\cdot\tilde{r})[p\,\omega(\lambda)\varepsilon(-\nu)\phi^*(\lambda-\nu)] \\
A_3(\lambda\nu) &= [p\,\omega(\lambda)\,r\,\phi^*(\lambda-\nu)][\tilde{r}\cdot\varepsilon(-\nu)] + [p\,\varepsilon(-\nu)\,r\,\phi^*(\lambda-\nu)][\tilde{r}\cdot\omega(\lambda)]
\end{aligned} \tag{95}$$

The helicity-coupling amplitudes are given by

$$F^J_{\lambda\nu} = g_1 A_1(\lambda\nu) + g_2 A_2(\lambda\nu) + g_3 A_3(\lambda\nu) \tag{96}$$

so that, from (58)–(67),

$$\begin{aligned} F^{(1)}_{++} &= W\left(g_1 - \frac{2}{3} g_2 r^2\right) \\ F^{(1)}_{0+} &= W\gamma_s \left(g_1 + \frac{1}{3} g_2 r^2 - g_3 r^2\right) \\ F^{(1)}_{+0} &= W\gamma_\sigma \left(g_1 + \frac{1}{3} g_2 r^2 + g_3 r^2\right) \end{aligned} \tag{97}$$

evaluated in the J rest frame. Again, the imaginary part has been absorbed into arbitrary complex constants g_1, g_2 and g_3. If the factors W, γ_s and γ_σ are taken out and if the substitutions

$$G^{(1)}_{01} = \sqrt{6} g_1, \quad G^{(1)}_{21} = \frac{2}{\sqrt{3}} g_2 r^2, \quad G^{(1)}_{22} = 2 g_3 r^2 \tag{98}$$

are made, the $F^{(1)}$'s of (92) obtained in the ℓS-coupling scheme reduce to the $F^{(1)}$'s given above. The reasons for the energy factors are clear: they come from the z-component of the wave functions $\omega(0)$ and $\varepsilon(0)$, which have been evaluated in the J rest frame and not in their respective rest systems [see (54)].

B. $J/\psi \to f_1(1420) + \gamma$

Let 'J', 's' and 'σ' stand for the J/ψ, the $f_1(1420)$ and the γ. This decay involves two helicity-coupling amplitudes F^J_{++} and F^J_{0+}, as photons do not have zero helicities.

Although the photon wave functions $\varepsilon(\pm)$ cannot be coupled to $\omega(m)$ with Clebsch-Gordan coefficients because of absence of $\varepsilon(0)$, the intermediate wave functions $\chi(m)$ [see (36)] and $\tau(m)$ could be considered as those corresponding to massive spin-1 particles, again with zero time components in the J rest frame. From this point of view, it is quite proper to introduce the concept of a pure intrinsic spin S and a pure orbital angular momentum ℓ in the decay process. One may then take over the results of the previous sub-section, remembering that gauge invariance is automatic in the present formulation.

Noting that $\tilde{r} \cdot \varepsilon(\pm) = 0$ for a photon, one obtains, from (95),

$$\begin{aligned} A_1(\lambda\nu) &= [p\,\omega(\lambda)\varepsilon(-\nu)\phi^*(\lambda-\nu)] \\ A_2(\lambda\nu) &= -[p\,\varepsilon(-\nu)\,r\,\phi^*(\lambda-\nu)][\tilde{r}\cdot\omega(\lambda)] \\ &\qquad - \frac{2}{3}(r\cdot\tilde{r})[p\,\omega(\lambda)\varepsilon(-\nu)\phi^*(\lambda-\nu)] \\ A_3(\lambda\nu) &= [p\,\varepsilon(-\nu)\,r\,\phi^*(\lambda-\nu)][\tilde{r}\cdot\omega(\lambda)] \end{aligned} \tag{99}$$

with $\nu = \pm 1$. This shows that A_2 is a superposition of A_1 and A_3 and that there are just two independent amplitudes corresponding to two F's in the problem. But the proportionality constant depends on r^2, a characteristic of $\ell = 2$ in the problem. By examining the first two equations of (97), one may write

$$\begin{aligned} F^{(1)}_{++} &= W(g_1 + g_2 r^2) \\ F^{(1)}_{0+} &= W\gamma_s(g_1 + g_3 r^2) \end{aligned} \tag{100}$$

with a slight change in the definition of g_i's. If one had used A_1 and A_3 only, the term with g_2 above would have been missing; one has adopted here a model in which both F^J's are made to depend on r^2, with the consequence that there now exist three unknowns g_1, g_2 and g_3 in the problem. It should be noted that the amplitude (94) would have given r^2 dependence to both F^J's.

The results are similar to those of Ref. [1], but not identical. The reason for this is that one has used $\tilde{g}(W)$ as the metric in this note, even in the presence of photons.

C. $\bar{p}p(^3S_1) \to K^*(892)\bar{K}^*(892)$

This decay process, $1^- \to 1^- + 1^-$, involves four helicity-coupling amplitudes and odd-ℓ orbital angular momenta ($\ell = 1$ or 3). They are given by, from (8),

$$\begin{aligned} \sqrt{2}F^{(1)}_{++} &= \sqrt{\frac{2}{3}}G^{(1)}_{10} - \sqrt{\frac{2}{15}}G^{(1)}_{12} + \frac{1}{\sqrt{5}}G^{(1)}_{32} \\ \sqrt{2}F^{(1)}_{0+} &= \frac{1}{\sqrt{2}}G^{(1)}_{11} - \sqrt{\frac{3}{10}}G^{(1)}_{12} - \frac{1}{\sqrt{5}}G^{(1)}_{32} \\ \sqrt{2}F^{(1)}_{+0} &= -\frac{1}{\sqrt{2}}G^{(1)}_{11} - \sqrt{\frac{3}{10}}G^{(1)}_{12} - \frac{1}{\sqrt{5}}G^{(1)}_{32} \\ F^{(1)}_{00} &= -\frac{1}{\sqrt{3}}G^{(1)}_{10} - \frac{2}{\sqrt{15}}G^{(1)}_{12} + \sqrt{\frac{2}{5}}G^{(1)}_{32} \end{aligned} \tag{101}$$

Again, it is seen that these equations satisfy (9).

Covariant amplitudes corresponding to pure P- and F-waves and pure total intrinsic spins 0, 1 and 2 are

$$\begin{aligned} A_1 &= (r \cdot \psi^{(0)} \cdot \phi^*) \\ A_2 &= (r \cdot \psi^{(1)} \cdot \phi^*) \\ A_3 &= (r \cdot \psi^{(2)} \cdot \phi^*) \\ A_4 &= \frac{1}{2}(\psi^{(2)} : \tilde{t}^{(3)} \cdot \phi^*) \end{aligned} \tag{102}$$

where the colon signifies contraction of two neighboring indices. The A_i's may be expressed as

$$
\begin{aligned}
A_1(\lambda\nu) &= [\tilde{\omega}(\lambda)\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)] \\
A_2(\lambda\nu) &= [\tilde{r}\cdot\omega(\lambda)][\tilde{\varepsilon}(-\nu)\cdot\phi^*(\lambda-\nu)] - [\tilde{r}\cdot\varepsilon(-\nu)][\tilde{\omega}(\lambda)\cdot\phi^*(\lambda-\nu)] \\
A_3(\lambda\nu) &= [\tilde{r}\cdot\omega(\lambda)][\tilde{\varepsilon}(-\nu)\cdot\phi^*(\lambda-\nu)] + [\tilde{r}\cdot\varepsilon(-\nu)][\tilde{\omega}(\lambda)\cdot\phi^*(\lambda-\nu)] \\
&\qquad -\frac{2}{3}[\tilde{\omega}(\lambda)\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)] \\
A_4(\lambda\nu) &= [\tilde{r}\cdot\omega(\lambda)][\tilde{r}\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)] \\
&\quad -\frac{1}{5}(\tilde{r}\cdot r)\Big\{[\tilde{\omega}(\lambda)\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)] + [\tilde{r}\cdot\varepsilon(-\nu)][\tilde{\omega}(\lambda)\cdot\phi^*(\lambda-\nu)] \\
&\qquad + [\tilde{r}\cdot\omega(\lambda)][\tilde{\varepsilon}(-\nu)\cdot\phi^*(\lambda-\nu)]\Big\}
\end{aligned}
\tag{103}
$$

In the J rest frame all the scalars in these amplitudes are reduced to those involving 3-momenta only. The helicity-coupling amplitudes are given by

$$
F^J_{\lambda\nu} = g_1A_1(\lambda\nu) + g_2A_2(\lambda\nu) + g_3A_3(\lambda\nu) + g_4A_4(\lambda\nu)
\tag{104}
$$

so that, from (58)–(63),

$$
\begin{aligned}
F^{(1)}_{++} &= \left(-g_1 + \frac{2}{3}g_3 + \frac{1}{5}g_4r^2\right) r \\
F^{(1)}_{0+} &= \gamma_s\left(g_2 + g_3 - \frac{1}{5}g_4r^2\right) r \\
F^{(1)}_{+0} &= \gamma_\sigma\left(-g_2 + g_3 - \frac{1}{5}g_4r^2\right) r \\
F^{(1)}_{00} &= \gamma_s\gamma_\sigma\left(g_1 + \frac{4}{3}g_3 + \frac{2}{5}g_4r^2\right) r
\end{aligned}
\tag{105}
$$

evaluated in the J rest frame. Aside from the energy dependent factors, these helicity-coupling amplitudes are seen to be identical to (101) with the following substitutions:

$$
G^{(1)}_{10} = -\sqrt{3}g_1r, \quad G^{(1)}_{11} = 2g_2r, \quad G^{(1)}_{12} = -2\sqrt{\frac{5}{3}}g_3r, \quad G^{(1)}_{32} = \sqrt{\frac{2}{5}}g_4r^3
\tag{106}
$$

D. $J/\psi \to \hat{\rho}(1405) + \gamma$

This decay mode, so far unobserved, may be considered a special case of the preceeding example, in the sense that the second decay product is massless, i.e. $\mu = 0$. Once again, one takes an approach of introducing the concept of a pure S and ℓ and take over the results of the previous sub-section with the condition $[\tilde{r}\cdot\varepsilon(\pm)] = 0$.

One finds, from (103),

$$
\begin{aligned}
A_1(\lambda\nu) &= [\tilde{\omega}(\lambda)\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)] \\
A_2(\lambda\nu) &= [\tilde{r}\cdot\omega(\lambda)][\tilde{\varepsilon}(-\nu)\cdot\phi^*(\lambda-\nu)]
\end{aligned}
\tag{107}
$$

$$A_3(\lambda\nu) = [\tilde{r}\cdot\omega(\lambda)][\tilde{\varepsilon}(-\nu)\cdot\phi^*(\lambda-\nu)] \\ \qquad -\frac{2}{3}[\tilde{\omega}(\lambda)\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)]$$

$$A_4(\lambda\nu) = -\frac{1}{5}(\tilde{r}\cdot r)\Big\{[\tilde{\omega}(\lambda)\cdot\varepsilon(-\nu)][\tilde{r}\cdot\phi^*(\lambda-\nu)] + [\tilde{r}\cdot\omega(\lambda)][\tilde{\varepsilon}(-\nu)\cdot\phi^*(\lambda-\nu)]\Big\}$$

with $\nu = \pm 1$. It is seen that A_3 and A_4 are superpositions of A_1 and A_2, but the proportionality constant for A_4 depends on r^2, a signature of an increase in ℓ by 2. One obtains, by setting $g_3 = 0$ in (105),

$$F^{(1)}_{++} = (g_1 + g_4 r^2) r \\ F^{(1)}_{0+} = \gamma_s (g_2 - g_4 r^2) r \tag{108}$$

with slight changes in g_i's. F^J's now depend on three unknowns, reflecting presence of $\ell = 3$ in the problem. Clearly, this result is more general than that presented in Ref. [1].

VII. SPIN-2 $\rightarrow$ SPIN-1 + SPIN-0

One may use, for this problem, the same notations and symbols introduced in Section 4. Thus, the 'J', 's' and 'σ' stand for the spin-2, spin-1 and spin-0 particles, respectively. As before, the treatment of this decay can be grouped into two separate categories depending on the intrinsic parities involved.

A. Net Intrinsic Parity $\eta_J \eta_s \eta_\sigma = +1$

An example of this case is provided by the decay $a_2(1320) \rightarrow \rho + \pi$. There exists only one helicity amplitude $F^{(2)}_+$ corresponding the D-wave allowed in the decay. Then, the covariant amplitude may be written, for the vectors defined along the z-axis,

$$A(m) = [p\ \omega(m)\ \tilde{t}^{(2)}\cdot\phi^*(m)] \\ = -\sum_{m_1 m_2} (1m_1\, 1m_2|2m)[\tilde{r}\cdot\phi^*(m_1)][p\ \omega(m)\ r\ \phi^*(m_2)] \tag{109}$$

where one has used the spin-2 wave function (25). The only non-zero component of A is

$$A(+) = -\frac{1}{\sqrt{2}}[\tilde{r}\cdot\phi^*(0)][p\ \omega(+)\ r\ \phi^*(+)] \tag{110}$$

to be evaluated in the J rest frame. From (59) and (62) one obtains for the helicity-coupling amplitude

$$F^{(2)}_+ = gWr^2 \tag{111}$$

where g is an arbitrary complex constant.

Suppose now that the particle s is a photon. Indeed, such a decay has been observed in $a_2(1320) \to \pi + \gamma$. Gauge invariance requires that one must replace ω according to the prescription given in (50), but the amplitude (110) does not change. The helicity-coupling amplitude (111) is thus appropriate for photons as well.

B. Net Intrinsic Parity $\eta_J \eta_s \eta_\sigma = -1$

The decay $\pi_2(1670) \to \rho + \pi$ is an example. There are two orbital angular momenta $\ell = 1$ and $\ell = 3$. The helicity-coupling amplitudes are

$$\begin{aligned} \sqrt{2} F_+^{(2)} &= \sqrt{\frac{3}{5}} G_1^{(2)} + \sqrt{\frac{2}{5}} G_3^{(2)} \\ F_0^{(2)} &= \sqrt{\frac{2}{5}} G_1^{(2)} - \sqrt{\frac{3}{5}} G_3^{(2)} \end{aligned} \tag{112}$$

The covariant amplitudes corresponding to G_1 and G_3 are

$$\begin{aligned} A_1 &= (\tilde{\omega} \cdot \phi^* \cdot \tilde{r}) \\ A_2 &= (\omega \cdot \tilde{t}^{(3)} : \phi^*) \end{aligned} \tag{113}$$

for pure P- and F-waves, respectively. In the J rest frame, they assume the form

$$\begin{aligned} A_1(m) &= \sum_{m_1 m_2} (1m_1\, 1m_2|2m)[\boldsymbol{\omega}(m) \cdot \boldsymbol{\phi}^*(m_1)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(m_2)] \\ &= (1m\, 10|2m)[\boldsymbol{\omega}(m) \cdot \boldsymbol{\phi}^*(m)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(0)] \\ A_2(m) &= \sum_{m_1 m_2} (1m_1\, 1m_2|2m)\Big\{[\boldsymbol{r} \cdot \boldsymbol{\omega}(m)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(m_1)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(m_2)] \\ &\qquad - \frac{2}{5} r^2 [\boldsymbol{\omega}(m) \cdot \boldsymbol{\phi}^*(m_1)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(m_2)]\Big\} \\ &= \delta_{m0}\, (10\, 10|20)[\boldsymbol{r} \cdot \boldsymbol{\omega}(0)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(0)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(0)] \\ &\quad - \frac{2}{5} r^2\, (1m\, 10|2m)[\boldsymbol{\omega}(m) \cdot \boldsymbol{\phi}^*(m)][\boldsymbol{r} \cdot \boldsymbol{\phi}^*(0)] \end{aligned} \tag{114}$$

From this one obtains

$$\begin{aligned} F_+^{(2)} &= \frac{1}{\sqrt{2}} \left(g_1 - \frac{2}{5} g_2 r^2 \right) r \\ F_0^{(2)} &= \sqrt{\frac{2}{3}} \gamma_s \left(g_1 + \frac{3}{5} g_2 r^2 \right) r \end{aligned} \tag{115}$$

Note that, if $\gamma_s = 1$, one can make the equations above identical to those in (112) with the following substitutions:

$$G_1^{(2)} = \sqrt{\frac{5}{3}} g_1 r, \quad G_3^{(2)} = -\sqrt{\frac{2}{5}} g_2 r^3 \tag{116}$$

The energy factor which breaks this symmetry is a necessary consequence of the fact that the wave function ω is being evaluated in the J rest system.

VIII. SPIN-1 $\rightarrow$ SPIN-2 + SPIN-0

As before, one may use the notations of Section 4: 'J', 's' and 'σ' stand for spin-1, spin-2 and spin-0, respectively.

A. Net Intrinsic Parity $\eta_J \eta_s \eta_\sigma = +1$

An example of this type of decay would be $\bar{p}p(^3S_1) \rightarrow a_2(1320) + \pi$. There is just one helicity-coupling amplitude $F_+^{(1)}$ corresponding to $\ell = 2$. Using the spin-2 wave function

$$\omega_{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1\ 1m_2|2m)\omega_\alpha(m_1)\omega_\beta(m_2) \tag{117}$$

or

$$\begin{aligned}
\omega_{\alpha\beta}(+2) &= \omega_\alpha(+)\,\omega_\beta(+) \\
\omega_{\alpha\beta}(+1) &= \frac{1}{\sqrt{2}}\Big[\omega_\alpha(+)\,\omega_\beta(0) + \omega_\alpha(0)\,\omega_\beta(+)\Big] \\
\omega_{\alpha\beta}(0) &= \frac{1}{\sqrt{6}}\Big[\omega_\alpha(+)\,\omega_\beta(-) + \omega_\alpha(-)\,\omega_\beta(+)\Big] + \sqrt{\frac{2}{3}}\omega_\alpha(0)\,\omega_\beta(0)
\end{aligned} \tag{118}$$

The covariant amplitude is

$$A(+) = \frac{1}{\sqrt{2}}[\tilde{r} \cdot \omega(0)]\,[p\,\omega(+)\,r\,\phi^*(+)] \tag{119}$$

The helicity-coupling amplitude is then

$$F_+^{(1)} = gW\gamma_s r^2 \tag{120}$$

for an arbitrary complex constant g.

B. Net Intrinsic Parity $\eta_J \eta_s \eta_\sigma = -1$

This case is exemplified by the process $\bar{p}p(^3P_1) \rightarrow f_2(1270) + \pi$. There are two helicity-coupling amplitudes corresponding to two orbital angular momenta $\ell = 1$ and $\ell = 3$:

$$\begin{aligned}
\sqrt{2}F_+^{(1)} &= -\sqrt{\frac{3}{5}}G_1^{(1)} - \sqrt{\frac{2}{5}}G_3^{(1)} \\
F_0^{(1)} &= -\sqrt{\frac{2}{5}}G_1^{(1)} + \sqrt{\frac{3}{5}}G_3^{(1)}
\end{aligned} \tag{121}$$

The covariant amplitudes are

$$A_1 = (r \cdot \tilde{\omega} \cdot \phi^*)$$
$$A_2 = (\omega : \tilde{t}^{(3)} \cdot \phi^*) \tag{122}$$

which are transformed into

$$\begin{aligned} A_1(m) &= \sum_{m_1 m_2} (1m_1\, 1m_2|2m)[\boldsymbol{r}\cdot\boldsymbol{\omega}(m_1)][\boldsymbol{\omega}(m_2)\cdot\boldsymbol{\phi}^*(m)] \\ A_2(m) &= \sum_{m_1 m_2} (1m_1\, 1m_2|2m)\Big\{[\boldsymbol{r}\cdot\boldsymbol{\omega}(m_1)][\boldsymbol{r}\cdot\boldsymbol{\omega}(m_2)][\boldsymbol{r}\cdot\boldsymbol{\phi}^*(m)] \\ &\quad -\frac{1}{5}r^2[\boldsymbol{\omega}(m_1)\cdot\boldsymbol{\omega}(m_2)][\boldsymbol{r}\cdot\boldsymbol{\phi}^*(m)] - \frac{2}{5}r^2[\boldsymbol{r}\cdot\boldsymbol{\omega}(m_1)][\boldsymbol{\omega}(m_2)\cdot\boldsymbol{\phi}^*(m)]\Big\} \end{aligned} \tag{123}$$

evaluated in the J rest frame. The helicity-coupling amplitudes are

$$\begin{aligned} F_+^{(1)} &= \frac{1}{\sqrt{2}}\gamma_s\left(g_1 - \frac{2}{5}g_2 r^2\right) r \\ F_0^{(1)} &= \sqrt{\frac{2}{3}}\Big[g_1\gamma_s^2 + \frac{3}{5}g_2\left(\frac{2}{3}\gamma_s^2 + \frac{1}{3}\right) r^2\Big] r \end{aligned} \tag{124}$$

Note that, as a result of the spin-2 character of the s, one finds multiple factors of γ_s in $F^{(1)}$. In the limit $\gamma_s \to 1$, the substitution

$$G_1^{(1)} = -\sqrt{\frac{5}{3}}g_1 r, \quad G_3^{(1)} = \sqrt{\frac{2}{5}}g_2 r^3 \tag{125}$$

into (121) leads to the results given above.

IX. SPIN-2 → SPIN-2 + SPIN-0

There are in total five orbital angular momenta, from $\ell = 0$ to $\ell = 4$. The problem again divides into two distinct cases.

A. Net Intrinsic Parity $\eta_J\eta_s\eta_\sigma = -1$

An example of this case is provided by the decay $\bar{p}p(^3P_2) \to f_2(1270) + \pi$. There are two helicity-coupling amplitudes $F_2^{(2)}$ and $F_1^{(2)}$ corresponding to $\ell = 1$ and $\ell = 3$:

$$\begin{aligned} \sqrt{2}F_2^{(2)} &= -\frac{2}{\sqrt{5}}G_1^{(2)} - \frac{1}{\sqrt{5}}G_3^{(2)} \\ \sqrt{2}F_1^{(2)} &= -\frac{1}{\sqrt{5}}G_1^{(2)} + \frac{2}{\sqrt{5}}G_3^{(2)} \end{aligned} \tag{126}$$

The $G^{(2)}$'s are the ℓS-coupling amplitudes.

The covariant amplitudes corresponding to pure orbital angular momenta are

$$\begin{aligned} A_1 &= (p\ r\ \omega \cdot \tilde{g} \cdot \phi^*) \\ A_2 &= (p\ \omega \cdot \tilde{t}^{(3)} \cdot \phi^*) \\ &= (p\ r\ \tilde{r} \cdot \omega\ \tilde{r} \cdot \phi^*) - \frac{1}{5}(r \cdot \tilde{r})(p\ r\ \omega \cdot \tilde{g} \cdot \phi^*) \end{aligned} \tag{127}$$

Using the spin-2 tensors given by (25) and (117), one finds

$$\begin{aligned} A_1(m) = &- \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} (1m_1\ 1m_2|2m)(1m_1'\ 1m_2'|2m) \\ &\qquad [\tilde{\omega}(m_1) \cdot \phi^*(m_1')][p\ \omega(m_2)\ r\ \phi^*(m_2')] \\ A_2(m) = &- \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} (1m_1\ 1m_2|2m)(1m_1'\ 1m_2'|2m) \\ &\qquad \left\{[\tilde{r} \cdot \omega(m_1)][\tilde{r} \cdot \phi^*(m_1')] - \frac{1}{5}(r \cdot \tilde{r})[\tilde{\omega}(m_1) \cdot \phi^*(m_1')]\right\} \\ &\qquad [p\ \omega(m_2)\ r\ \phi^*(m_2')] \end{aligned} \tag{128}$$

Note that $A_1(0) = A_2(0) = 0$. The amplitudes with $m = +2$ and $m = +1$ lead to

$$\begin{aligned} F_2^{(2)} &= W\left(g_1 - \frac{1}{5}g_2 r^2\right) r \\ F_1^{(2)} &= W\gamma_s\left(\frac{1}{2}g_1 + \frac{2}{5}g_2 r^2\right) r \end{aligned} \tag{129}$$

for two arbitrary complex constants g_1 and g_2. Again, neglecting the mass and energy factors, the substitutions

$$G_1^{(2)} = -\sqrt{\frac{5}{2}}g_1 r, \quad G_3^{(2)} = \sqrt{\frac{2}{5}}g_2 r^3 \tag{130}$$

into (126) lead to the results given above.

B. Net Intrinsic Parity $\eta_J \eta_s \eta_\sigma = +1$

An example for this case is $\pi_2(1670) \to f_2(1270) + \pi$. There are three helicity-coupling amplitudes corresponding to $\ell = 0$, $\ell = 2$ and $\ell = 4$, which are given by

$$\begin{aligned} \sqrt{2}F_2^{(2)} &= \sqrt{\frac{2}{5}}G_0^{(2)} + \frac{2}{\sqrt{7}}G_2^{(2)} + \frac{1}{\sqrt{35}}G_4^{(2)} \\ \sqrt{2}F_1^{(2)} &= \sqrt{\frac{2}{5}}G_0^{(2)} - \frac{1}{\sqrt{7}}G_2^{(2)} - \frac{4}{\sqrt{35}}G_4^{(2)} \\ F_0^{(2)} &= \frac{1}{\sqrt{5}}G_0^{(2)} - \sqrt{\frac{2}{7}}G_2^{(2)} + 3\sqrt{\frac{2}{35}}G_4^{(2)} \end{aligned} \tag{131}$$

in terms of the ℓS-coupling amplitudes.

The covariant amplitudes corresponding to pure orbital angular momenta are

$$
\begin{aligned}
A_1 &= (\tilde{\omega} \ : \phi^*) \\
A_2 &= (\cdot\ \omega \cdot \tilde{t}^{(2)} \cdot \phi^* \cdot)_W \\
&= (\tilde{r} \cdot \omega \cdot \tilde{g} \cdot \phi^* \cdot \tilde{r}) - \frac{1}{3}(r \cdot \tilde{r})(\tilde{\omega} \ : \phi^*) \\
A_3 &= (\omega : \tilde{t}^{(4)} : \phi^*) \\
&= (\tilde{r} \cdot \omega \cdot \tilde{r})(\tilde{r} \cdot \phi^* \cdot \tilde{r}) - \frac{4}{7}(r \cdot \tilde{r})(\tilde{r} \cdot \omega \cdot \tilde{g} \cdot \phi^* \cdot \tilde{r}) \\
&\quad -\frac{1}{7}(r \cdot \tilde{r})(\cdot\ \omega\ \cdot)_W(\tilde{r} \cdot \phi^* \cdot \tilde{r}) + \frac{2}{35}(r \cdot \tilde{r})^2(\tilde{\omega} \ : \phi^*)
\end{aligned}
\tag{132}
$$

where the notation $(\cdot \quad \cdot)_W$ signifies that two remaining indices within the parenthesis, the first and the last, are to be contracted with the metric $\tilde{g}(W)$. Using (117), one may write

$$
\begin{aligned}
A_1(m) = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} & (1m_1\ 1m_2|2m)(1m_1'\ 1m_2'|2m) \\
& [\tilde{\omega}(m_1) \cdot \phi^*(m_1')][\tilde{\omega}(m_2) \cdot \phi^*(m_2')] \\
A_2(m) = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} & (1m_1\ 1m_2|2m)(1m_1'\ 1m_2'|2m) \\
& \Big\{[\omega(m_1) \cdot \phi^*(m_1')][\tilde{r} \cdot \omega(m_1)][\tilde{r} \cdot \phi^*(m_1')] \\
& \quad - \frac{1}{3}(r \cdot \tilde{r})[\tilde{\omega}(m_1) \cdot \phi^*(m_1')][\tilde{\omega}(m_2) \cdot \phi^*(m_2')]\Big\} \\
A_3(m) = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} & (1m_1\ 1m_2|2m)(1m_1'\ 1m_2'|2m) \\
& \Big\{[\tilde{r} \cdot \omega(m_1)][\tilde{r} \cdot \phi^*(m_1')][\tilde{r} \cdot \omega(m_2)][\tilde{r} \cdot \phi^*(m_2')] \\
& \quad - \frac{4}{7}(r \cdot \tilde{r})[\omega(m_1) \cdot \phi^*(m_1')][\tilde{r} \cdot \omega(m_1)][\tilde{r} \cdot \phi^*(m_1')] \\
& \quad - \frac{1}{7}(r \cdot \tilde{r})[\tilde{\omega}(m_1) \cdot \omega(m_2)][\tilde{r} \cdot \phi^*(m_1')][\tilde{r} \cdot \phi^*(m_2')] \\
& \quad + \frac{2}{35}(r \cdot \tilde{r})^2[\tilde{\omega}(m_1) \cdot \phi^*(m_1')][\tilde{\omega}(m_2) \cdot \phi^*(m_2')]\Big\}
\end{aligned}
\tag{133}
$$

These lead to the following form for the helicity-coupling amplitudes

$$
\begin{aligned}
F_2^{(2)} &= g_1 - \frac{1}{3}g_2 r^2 + \frac{2}{35}g_3 r^4 \\
F_1^{(2)} &= \gamma_s \left(g_1 + \frac{1}{6}g_2 r^2 - \frac{8}{35}g_3 r^4\right) \\
F_0^{(2)} &= g_1 \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3}\right) + \frac{1}{3}g_2 \left(\frac{4}{3}\gamma_s^2 - \frac{1}{3}\right) r^2 + \frac{12}{35}g_3 \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3}\right) r^4
\end{aligned}
\tag{134}
$$

where g_1, g_2 and g_3 are three arbitrary complex constants.

If and only if one sets $\gamma_s = 1$, these formula are seen to be identical to (131) after the substitutions

$$G_0^{(2)} = \sqrt{5} g_1, \quad G_2^{(2)} = -\frac{1}{3}\sqrt{\frac{7}{2}} g_2 r^2, \quad G_4^{(2)} = 2\sqrt{\frac{2}{35}} g_3 r^4 \tag{135}$$

X. SPIN-0 → SPIN-2 + SPIN-0

The decay $\eta_c(2980) \to a_2(1230) + \pi$ provides an example for this process. There is one helicity-coupling amplitude for $\ell = 2$ and $\eta_J \eta_s \eta_\sigma = +1$ always. The covariant amplitude is

$$A = (\tilde{t}^{(2)} : \omega) = (\tilde{r} \cdot \omega \cdot \tilde{r}) - \frac{1}{3}(r \cdot \tilde{r})(\cdot\ \omega\ \cdot)_W \tag{136}$$

for a pure D-wave, which can be transformed into, from (72) and (118),

$$A(0) = \sqrt{\frac{2}{3}} \gamma_s^2 r^2 - \frac{1}{3}\sqrt{\frac{2}{3}} (\gamma_s^2 - 1) r^2 \tag{137}$$

The helicity-coupling amplitude is

$$F_0^{(0)} = \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3}\right) g\, r^2 \tag{138}$$

where g is a complex constant.

XI. SPIN-3 → SPIN-1 + SPIN-0

In this section one deals with a $J = 3$ state coupling to a $s = 1$ or $s = 2$ state and a pseudoscalar.

A. $\phi_3(1850) \to K^*(892)\bar{K}$

There exists just one F_+^J corresponding to $\ell = 3$. The covariant amplitude may be written

$$A = [p\, \omega(+)\, \tilde{t}^{(3)} : \phi^*(+)] \tag{139}$$

The only non-zero term in A is, from (32),

$$A = \frac{2}{\sqrt{15}} [\tilde{r} \cdot \phi^*(0)]^2\ [p\, \omega(+)\, r\, \phi^*(+)] \tag{140}$$

From this one finds

$$F_+^J = g W r^3 \tag{141}$$

for an arbitrary constant g.

B. $\omega_3(1670) \to b_1(1235)\pi$

There are two F^J's corresponding to $\ell = 2$ and $\ell = 4$:

$$
\begin{aligned}
\sqrt{2}F_+^J &= \frac{2}{\sqrt{7}}G_2^J + \sqrt{\frac{3}{7}}G_4^J \\
F_0^J &= \sqrt{\frac{3}{7}}G_2^J - \frac{2}{\sqrt{7}}G_4^J
\end{aligned}
\tag{142}
$$

The covariant amplitudes corresponding to $\ell = 2$ and $\ell = 4$ are

$$
\begin{aligned}
A_2 &= (\tilde{t}^{(2)} : \phi^* \cdot \tilde{\omega}) \\
A_4 &= (\omega \cdot \tilde{t}^{(4)} \therefore \phi^*)
\end{aligned}
\tag{143}
$$

where '$\therefore$' signifies contraction over three neighboring indices. With the aid of (32), one obtains

$$
\begin{aligned}
F_+^J &= \frac{2}{\sqrt{15}}\left(g_2 - \frac{3}{7}g_4 r^2\right) r^2 \\
F_0^J &= \gamma_s \sqrt{\frac{2}{5}}\left(g_2 + \frac{4}{7}g_4 r^2\right) r^2
\end{aligned}
\tag{144}
$$

In the limit $\gamma_s \to 1$, the formulas (142) and (144) are seen to be identical with the substitutions

$$
G_2^J = \sqrt{\frac{14}{15}}g_2 \quad \text{and} \quad G_4^J = -2\sqrt{\frac{2}{35}}g_4
\tag{145}
$$

C. $a_3(2050) \to f_2(1270)\pi$

This decay is so far unobserved, but it affords an opportunity to explore new territory regarding the structure of helicity-coupling amplitudes. There are three F^J's corresponding to $\ell =$1, 3 and 5.

$$
\begin{aligned}
\sqrt{2}F_2^J &= \sqrt{\frac{2}{7}}G_1^J + \sqrt{\frac{2}{3}}G_3^J + \frac{1}{\sqrt{21}}G_5^J \\
\sqrt{2}F_1^J &= \frac{4}{\sqrt{35}}G_1^J - \frac{1}{\sqrt{15}}G_3^J - \sqrt{\frac{10}{21}}G_5^J \\
F_0^J &= \frac{3}{\sqrt{35}}G_1^J - \frac{2}{\sqrt{15}}G_3^J + \sqrt{\frac{10}{21}}G_5^J
\end{aligned}
\tag{146}
$$

The covariant amplitudes are

$$
\begin{aligned}
A_1(m) &= [\tilde{\omega}(m) : \phi^*(m) \cdot \tilde{r}] \\
A_3(m) &= [\cdot\, \omega(m) \cdot \tilde{t}^{(3)} : \phi^*(m) \cdot]_W \\
A_5(m) &= [\tilde{\omega}(m) : \tilde{t}^{(5)} \therefore \phi^*(m)]
\end{aligned}
\tag{147}
$$

for $\ell = 1$, $\ell = 3$ and $\ell = 5$, respectively. Again, the notation $[\cdot \quad \cdot]_W$ indicates that the first and the last free indices within $[\quad]$ are to be summed over with the modified metric $\tilde{g}(W)$. With the aid of (32), (45) and (118), one finds

$$
\begin{aligned}
F_2^J &= \frac{1}{\sqrt{3}}\ \left(g_1 - \frac{2}{5} g_3 r^2 + \frac{2}{21} g_5 4^4 \right) r \\
F_1^J &= \sqrt{\frac{2}{15}}\ \gamma_s\ \left(2g_1 + \frac{1}{5} g_3 r^2 - \frac{10}{21} g_5 r^4 \right) r \\
F_0^J &= \sqrt{\frac{3}{5}}\ \left\{ g_1\ \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) + \frac{4}{15} g_3\ \left(\frac{3}{2}\gamma_s^2 - \frac{1}{2} \right) r^2 + \frac{20}{63} g_5\ \left(\frac{2}{3}\gamma_s^2 + \frac{1}{3} \right) r^4 \right\} r
\end{aligned}
\tag{148}
$$

where g_1, g_3 and g_5 are arbitrary constants. If one takes the limit $\gamma_s \to 1$, then the following substitutions

$$
G_1^J = \sqrt{\frac{7}{3}}\ g_1, \qquad G_3^J = -\frac{2}{5}\ g_3 \quad \text{and} \quad G_5^J = 2\sqrt{\frac{2}{63}} g_5 \tag{149}
$$

transforms (146) into (148).

XII. CONCLUSIONS

In this note, a general formalism is developed for constructing covariant helicity-coupling amplitudes $F^J_{\lambda\nu}$ in an arbitrary two-body decay $J \to s + \sigma$. The decay amplitudes are given as expansions in the total intrinsic spin S and pure orbital angular momentum ℓ. For the purpose, one has introduced intermediate wave functions $\chi(m_s)$ and $\tau(m)$, which are the tensor analogues of the ket states $|Sm_s\rangle$ and $|\ell m\rangle$. By requiring that they have vanishing time components in the J rest frame, the covariant decay amplitudes reduce to those involving 3-vectors only in the J rest frame. This is a general rule without exception: disregard time components of all the four-vectors p, r, ω, ε, and ϕ^* in the problem; replace $\tilde{g}(W)$ by δ_{ij} with $i,j = 1,2,3$ wherever an inner product appears in the amplitudes; and replace the Lorentz scalars $(pabc)$ by their 3-vector counterpart $W(\boldsymbol{abc})$. This rule applies even to an S-wave decay, e.g. in $b_1(1235) \to \omega\pi$: the amplitude is $(\boldsymbol{\omega} \cdot \boldsymbol{\phi}^*)$ in the J rest frame.

The helicity coupling amplitudes have been shown to depend in general on four variables: W, r, γ_s and γ_σ, with the latter three evaluated in the J rest frame. Note that, if the states J, s and σ have finite widths, then the masses W, m and μ themselves are all continuous variables. In addition, it has been shown that the dependence on γ's is present only if the spins s and σ are greater than zero. Indeed, if $s = \sigma = 0$, then the covariant helicity-coupling amplitude F_0^J is merely given by r^J, identical to the result of non-relativistic formalism.

Energy dependence of the helicity-coupling amplitudes is a necessary consequence of the fact that the decay products s and σ have finite momenta in the J rest frame. Specifically, if one of the decay product 'σ' is a spin-0 particle, the helicity-coupling amplitude F^J_λ is in general a polynomial of order $s - |\lambda|$ in γ_s, where λ is the helicity of s. The functional dependence on γ_s is simple indeed if $s = 1$: it is a monomial. Thus F^J_0 is simply proportional to γ_s. Likewise, if $s = 2$, then F^J_1 is shown to be proportional to γ_s, but F^J_0 depends, for the examples treated in this note, on the following functional forms of γ_s:

$$\gamma_s^2, \quad \frac{2}{3}\gamma_s^2 + \frac{1}{3}, \quad \frac{4}{3}\gamma_s^2 - \frac{1}{3} \quad \text{and} \quad \frac{3}{2}\gamma_s^2 - \frac{1}{2} \tag{150}$$

Each is normalized to one as $\gamma_s \to 1$, i.e. in the non-relativistic limit.

In the limit $W \to \infty$, the variables r and γ's also go to ∞. The factor r^ℓ is in practice always replaced by a Blatt-Weisskopf function (see [1]) which approaches a constant as $r \to \infty$. In the absence of dynamics imposing functional forms, the undamped γ factors must remain as such in the covariant amplitudes. One sees then that the angular distribution, in the limit $W \to \infty$, is determined by only one F^J_{ab} where $a = \min|\lambda|$ and $b = \min|\nu|$ allowed in the problem.

It is advocated that the decays involving a photon in the final state be treated in the same way: the decay amplitudes are given an expansion in a state of definite S and ℓ, as if the photon were a massive vector particle, e.g. $\rho(770)$ or $\omega(782)$. One then imposes a condition that the photon wave function have no zero z-component. It must be emphasized that the intermediate wave functions χ and τ (see Section III), required for this procedure, have nothing to do with the photon; they correspond to those of unobserved (and massive) spin-1 or higher-spin particles, defined to have zero time components in the parent rest frame. This approach allows for photons to be treated in exactly the same way as massive particles—appropriate for helicity-coupling amplitudes.

The unknowns in the decay problem, denoted as g_i's in this note throughout, have been treated as constants. It should be clear, however, that one has chosen here a <u>model</u>—one which satisfies Lorentz invariance, incorporating the concept of definite S and ℓ. It is shown that there exists a one-to-one correspondence between the number of $F^J_{\lambda\nu}$'s and g_i's. This is certainly the case, without exceptions, for the decays involving massive particles. However, if one of the decay particles is a photon, then the number of g_i's can exceed that of $F^J_{\lambda\nu}$'s, as shown in two examples in this note. In a model in which g_i's are treated as constants, this is unavoidable. What one has done in fact is to suppose that g_i's can be given an expansion in r^2 (but not in r because of parity conservation) and then retain the first two terms.

It should be pointed out that the F^J_0's in (116), (125) and (128) of Ref. [1] are incorrect; the correct ones are given in this note in (124), (134) and (138), respectively. All these errors resulted from neglecting in the covariant amplitudes a term proportional to $[\tilde{g}(W) : \omega]$ given in (72). In Ref. [1], several amplitudes have been given summed over the z-components of spin for s and σ, in (2), (8), (51) and (56). They are misleading and hence dropped in this note, as these sums should always be performed along with the amplitudes corresponding to decays of s and σ [see (84)]. In addition, a number of sloppy notations and typographical errors in Ref. [1] have been corrected.

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The author acknowledges a recent paper by Filippini, Fontana and Rotondi [4] on covariant spin tensors in meson spectroscopy. They pointed out in their paper that the helicity-coupling amplitude of Section X was incorrect in the original paper by the author [1]. They also pointed out that, because of the γ factors, angular distributions for S-wave decays are no longer isotropic; this has been illustrated in this note with the decay $b_1(1235) \to \omega\pi$ (see Section VA).

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