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C- and G-parity: a New Definition and Applications —Version III—

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abstract

A new definition for C (charge-conjugation) and G operations is proposed which allows for unique value of the C parity for each member of a given J^{PC} nonet. A simple straightforward extension of the definition allows quarks to be treated on an equal footing. As illustrative examples, the problems of constructing eigenstates of C, I and G operators are worked out for $\pi\pi$, $K\bar{K}$, $N\bar{N}$ and $q\bar{q}$ systems. In particular, as an illustration of the application to three-body systems, the case of a neutral $K\bar{K}\pi$ system is worked out in some detail.

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1 Introduction

The purpose of this note is to point out that the C operation can be defined in such a way that a unique value can be assigned to all the members of a given J^{PC} nonet. In conventional treatments in which antiparticle states are defined through C, one encounters the problem that anti-particle states do not transform in the same way (the so-called charge-conjugate representation). That this is so is obvious if one considers the fact that a C operation changes sign of the z-component of the I-spin, so that in general C and I operators do not commute. In our treatment, the anti-particle states have been defined so that they transform under isospin rotations in exactly the same way as those of particle states. This allows for use of the same SU(2) Clebsch-Gordan coefficients for particle and anti-particle states, a distinctive advantage in phenomenological applications.

Arbitrary parameters in the problem have been carefully chosen so as to obtain conventional formulae for CP-eigenstates of neutral kaons K^0 and $\bar{K}^0$. Anti-quark states, conventionally, are defined such that they do not transform in the same way as those quark states. The relationship of our definition to this more conventional one is also pointed out.

As illustrative examples, states of particle-antiparticle systems are worked out, in which use is made of the symmetry under interchange of two wave functions (spin statistics). As an application to three-body systems, the case of a neutral $K\bar{K}\pi$ system is studied in some detail, in particular to point out several non-trivial results regarding the system.

2 G-and C-parity Operations

We shall adopt a notation 'a' to stand for both the baryon number B and hypercharge Y. Anti-particles are denoted '$\bar{a}$', so that

$$a = (B, Y), \quad \bar{a} = (\bar{B}, \bar{Y}) = (-B, -Y) \tag{1}$$

In addition, we shall use y to denote $Y/2$;

$$y = \frac{Y}{2} = \frac{1}{2}(B + S), \quad Q = y + \nu \tag{2}$$

where S, Q, ν are the strangeness, the charge and the third component of isospin, respectively.

We start with a state having an isospin σ and its third component ν which transforms according to the standard $|jm\rangle$ representation. Let I be the isospin operator. Then,

$$\begin{aligned} I_z|\sigma\nu\rangle &= \nu|\sigma\nu\rangle \\ I_\pm|\sigma\nu\rangle &= F_\pm(\nu)|\sigma\nu\pm 1\rangle \\ I^2|\sigma\nu\rangle &= \sigma(\sigma+1)|\sigma\nu\rangle \end{aligned} \tag{3}$$

where $I_\pm = I_x \pm iI_y$ and

$$F_\pm(\nu) = \sqrt{(\sigma \mp \nu)(\sigma \pm \nu + 1)} \tag{4}$$

Note that $F_\pm(\nu) = F_\mp(-\nu)$. We shall require that anti-particle states transform in the same way as their particle states according to (3).

The C operation changes a state $|a\nu\rangle$ to $|\bar{a}\ -\nu\rangle$. (We use a shorthand notation where the isospin σ is omitted from a more complete description of the state $|a\sigma\nu\rangle$.) If anti-particle states are to transform in the same way as particle states, it is necessary that one define an anti-particle through the G operation. The key point is that G is defined so that its operation does not perturb the ν quantum number. To define the G operator, we need to first introduce a rotation by 90° around the y-axis:

$$R_y(\pi)|\sigma\nu\rangle = (-)^{\sigma-\nu}|\sigma\ -\nu\rangle \tag{5}$$

It will be shown later that $R_y(\pi)$ commutes with the C operator. We therefore define the G operator through

$$G = CR_y(\pi) = R_y(\pi)C \tag{6}$$

We are now ready to define an anti-particle state via

$$\begin{aligned} G|a\nu\rangle &= g|\bar{a}\nu\rangle \\ G|\bar{a}\nu\rangle &= \bar{g}|a\nu\rangle \end{aligned} \tag{7}$$

and *require* that g and $\bar{g}$ be independent of ν and furthermore that an arbitrary isospin rotation R commutes with G:

$$[R, G] = 0 \tag{8}$$

The action of C on particle and anti-particle states is readily discerned through (5) and (6);

$$\begin{aligned} C|a\nu\rangle &= g(-)^{\sigma+\nu}|\bar{a}-\nu\rangle \\ C|\bar{a}\nu\rangle &= \bar{g}(-)^{\sigma+\nu}|a-\nu\rangle \end{aligned} \tag{9}$$

It is customary to define C such that $C^2 = +\,1$, in which case

$$g\bar{g}(-)^{2\sigma} = +1 \tag{10}$$

For hadrons, we shall define g and $\bar{g}$ via

$$g = \eta(-)^{y+\sigma}, \quad \bar{g} = \eta(-)^{\bar{y}+\sigma} \tag{11}$$

while for quarks,

$$g = \eta(-)^{B+y+\sigma}, \quad \bar{g} = \eta(-)^{\bar{B}+\bar{y}+\sigma} \tag{12}$$

Note that the exponents in these expressions are always integers. Note in particular that, for quarks, B and $\bar{B}$ are needed to make the exponents integers. The quantity F defined by

$$\frac{1}{2}F = B + y = \frac{1}{2}(3B + S)$$

will be termed the '*intrinsic flavor*' of a particle. Note that the intrinsic flavor is always an integer, as shown in the following table:

states	u	d	s	π	η	K	p	n	Λ	Σ	Ξ	Δ	Ω^-
F	1	1	0	0	0	1	3	3	2	2	1	3	0

It is seen that the intrinsic flavor of an anti-particle is the negative of that of the particle, i.e. $\bar{F} = -F$. With these definitions, we can make η a real number and let it take on values of $+1$ or-1, so that $\eta^2 = +1$. Then, we have,

$$C^2 = +1, \quad G^2 = (-)^{2\sigma} \tag{13}$$

conforming to the standard expressions.

We have adopted two different definitions for g—one for hadrons and another for quarks; however, we could have chosen a single convention in which the expression (12) is used for both hadrons and for quarks. The factor $(-)^B$ for mesons is $+1$, while it is -1 for baryons, but the extra minus sign could be simply absorbed into η. Although it might be aesthetically pleasing to have a single definition for g (along with the 'intrinsic flavor' F defined above), we choose to opt for more transparent definitions.

From (3) and (9), it is easy to work out the commutation relations between C and I;

$$\{C, I_x\} = \{C, I_z\} = 0, \quad [C, I_y] = 0 \tag{14}$$

ln other words, C anti-commutes with I_x and I_z while it commutes with I_y. This gives a ready justification of the definition of G-parity given in (6). From (8), we can further deduce that

$$CRC^{-1} = R_y(\pi)RR_y^{-1}(\pi) \tag{15}$$

This shows that the actions of I-spin rotation under charge-conjugation can be expressed in terms of I-spin 90° rotations.

We end this section by recapitulating the actions of G and C. For hadrons, we have, from (7), (9) and (11),

$$\begin{aligned} G|a\nu\rangle &= \eta(-)^{y+\sigma}|\bar{a}\nu\rangle \\ G|\bar{a}\nu\rangle &= \eta(-)^{\bar{y}+\sigma}|a\nu\rangle \end{aligned} \tag{16}$$

$$\begin{aligned} C|a\nu\rangle &= \eta(-)^{y-\nu}|\bar{a}\ \ -\nu\rangle \\ C|\bar{a}\nu\rangle &= \eta(-)^{\bar{y}-\nu}|a\ \ -\nu\rangle \end{aligned} \tag{17}$$

and, likewise, for quarks we have,

$$\begin{aligned} G|a\nu\rangle &= \eta(-)^{B+y+\sigma}|\bar{a}\nu\rangle \\ G|\bar{a}\nu\rangle &= \eta(-)^{\bar{B}+\bar{y}+\sigma}|a\nu\rangle \end{aligned} \tag{18}$$

$$\begin{aligned} C|a\nu\rangle &= \eta(-)^{B+y-\nu}|\bar{a}\ \ -\nu\rangle \\ C|\bar{a}\nu\rangle &= \eta(-)^{\bar{B}+\bar{y}-\nu}|a\ \ -\nu\rangle \end{aligned} \tag{19}$$

Note that $\bar{y} = -y$ and $\bar{B} = -B$. It is worth emphasizing again that all the exponents in (16) through (19) are integers. Note in addition that only the u and d quarks have I-spin flavors under strong interactions. So, in practice, (18) and (19) apply only to these quarks, although s and other heavier quarks, c, b and t, can also be described in the same fashion with $\sigma = 0$ and the strangeness S replaced by heavier flavors $C(= +1)$, $B(= -1)$ and $T(= +1)$.

3 Examples of Single-Particle States

Nonstrange neutral mesons have, of course, $a = \bar{a}$, $y = \bar{y} = 0$ and $\nu = 0$. The expression (17) shows that a state $|a\nu\rangle$ is in an eigenstate of C with the eigenvalue η. Nonstrange charged or neutral mesons are, according to (16), in an eigenstate of G with the eigenvalue $\eta(-)^\sigma$, a familiar result. Since the expressions (16) and (17) are general and applies to all hadrons, it is natural to extend to the strange members of a given J^{PC} nonet the same η which is determined for only the nonstrange neutral member of the family. It should be borne in mind, however, that the value of η cannot be directly determined for strange mesons.

As a first example, let us take the π nonet ($J^{PC} = 0^{-+}$) and exhibit the actions G and C. The η and π^0 mesons are in an eigenstate of C with the eigenvalue $+1$, so that the value of η in (16) and (17) is equal to $+1$. The actions of C and G for other members of the nonet follow:

$$\begin{array}{lcllcl} G|\pi\rangle & = & -|\pi\rangle, & C|\pi^\pm\rangle & = & -|\pi^\mp\rangle \\ G|K^+\rangle & = & -|\bar{K}^0\rangle, & C|K^+\rangle & = & +|K^-\rangle \\ G|K^0\rangle & = & -|K^-\rangle, & C|K^0\rangle & = & -|\bar{K}^0\rangle \\ G|\bar{K}^0\rangle & = & +|K^+\rangle, & C|\bar{K}^0\rangle & = & -|K^0\rangle \\ G|K^-\rangle & = & +|K^0\rangle, & C|K^-\rangle & = & +|K^+\rangle \end{array} \tag{20}$$

The eigenstates of CP for a neutral K system is now given by the familiar relations.

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0 + |\bar{K}^0\rangle) \tag{21}$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}(|K^0 - |\bar{K}^0\rangle)$$

where the physically-allowed neutral K states K_S and K_L are nearly equal to K_1 and K_2, respectively—but not identical—due to CP-violation in their decays.

The nonet of ground-state baryons containing nucleons can be defined in a similar way as in (20). Since the neutral members are no longer in an eigenstate of C or G, we arbitrarily set $\eta = +1$ for this nonet (or $\eta = -1$ if the intrinsic flavor is used):

$$G|p\rangle = -|\bar{n}\rangle, \qquad C|p\rangle = +|\bar{p}\rangle \tag{22}$$

$$\begin{array}{ll} G|n\rangle = -|\bar{p}\rangle, & C|n\rangle = -|\bar{n}\rangle \\ G|\bar{n}\rangle = +|p\rangle, & C|\bar{n}\rangle = -|n\rangle \\ G|\bar{p}\rangle = +|n\rangle, & C|\bar{p}\rangle = +|p\rangle \end{array} \tag{23}$$

Other members of the nonet, Λ, Ξ, Σ, can be worked out in a similar way.

From (18) and (19), it can be seen that the up- and down-quark states will transform in exactly the same way as in (22) and (23) if we again set $\eta = +1$; merely replace p and n in these formulae by u and d. We exhibit isospin 0 and 1 states for quark-antiquark systems:

$$
\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}(|u\rangle|\bar{u}\rangle - |d\rangle\bar{d}\rangle) \\
|10\rangle &= \frac{1}{\sqrt{2}}(|u\rangle|\bar{u} + |d\rangle|\bar{d}\rangle)
\end{aligned}
\tag{24}
$$

where we have used the usual Clebsch-Gordon Coefficient tables. However, it has now become customary to define antiquark states via C operations as follows,

$$
\begin{aligned}
C|u\rangle &= +|\bar{u}\rangle_c, \qquad & C|d\rangle &= +|\bar{d}\rangle_c \\
C|\bar{d}\rangle &= +|d\rangle_c, \qquad & C|\bar{u}\rangle &= +|u\rangle_c
\end{aligned}
\tag{25}
$$

where the subscripts c have been introduced to distinguish these from our definition of antiparticle states as given in (22) and (23). It is clear that the up-quark states transform in the same way, but the down-quark states differ by a minus sign. In terms of the more conventional definitions (25), the isospin 0 and 1 are now read

$$
\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}(|u\rangle_c|\bar{u}\rangle_c + |d\rangle_c|\bar{d}\rangle_c) \\
|10\rangle &= \frac{1}{\sqrt{2}}(|u\rangle_c|\bar{u}\rangle_c - |d\rangle_c|\bar{d}\rangle_c)
\end{aligned}
\tag{26}
$$

It is in these forms that one encounters most often the isospin states of quark-antiquark systems. One must not forget, however, that these antiquark states do not have the 'correct' isospin rotation properties.

4 Two-Particle States

We shall work out here the effect of C and G operations on a particle-antiparticle system in an eigenstate of total isospin, total intrinsic spin, orbital angular momentum and total spin. We use the notations I, S, ℓ and J for these quantum numbers. (Note that I was used as an isospin operator and S denoted strangeness in section2.) Each single-particle state in the two-particle center-of-mass(CM) system will be given a shorthand notation,

$$
\begin{aligned}
|a, +\vec{k}, \nu_1\rangle &= |a, +\vec{k}, \sigma_1\nu_1, s_1 m_1\rangle \\
|\bar{a}, -\vec{k}, \nu_2\rangle &= |\bar{a}, -\vec{k}, \sigma_2\nu_2, s_2 m_2\rangle
\end{aligned}
\tag{27}
$$

where $\vec{k}$ is the 3-momentum of the particle in the CM system, and σ_1 and s_1 are isospin and spin of the particles $\sigma_1 = \sigma_2 = \sigma$ and $s_1 = s_2 = s$.

The two-particle system in a given state of $|I\nu\rangle$ and $|\ell SJM\rangle$ is given by

$$\begin{aligned} |a\bar{a}\nu\rangle &= \sum(\sigma_1\nu_1\sigma_2\nu_2|I\nu) \\ & \quad (s_1m_1s_2m_2|Sm_s)(Sm_s\ell m|JM)\int \mathrm{d}\vec{k}Y_{\ell m}(\vec{k}) \\ & \quad |a,+\vec{k},\nu_1\rangle|\bar{a},-\vec{k},\nu_2\rangle \end{aligned} \tag{28}$$

where Y_{lm} is the usual spherical harmonics. We note, from (16) through (19),

$$\begin{aligned} C|a,+\vec{k},\nu_1\rangle|\bar{a},-\vec{k},\nu_2\rangle &= (-)^{\nu_1+\nu_2}|\bar{a},+\vec{k},-\nu_1\rangle|a,-\vec{k},\ -\nu_2\rangle \\ &= (-)^{\nu_1+\nu_2+2s}|a,-\vec{k},\ -\nu_2\rangle|\bar{a},+\vec{k},\ -\nu_1\rangle \end{aligned} \tag{29}$$

$$\begin{aligned} G|a,+\vec{k},\nu_1\rangle|\bar{a},-\vec{k},\nu_2\rangle &= (-)^{2\sigma}|\bar{a},+\vec{k},\nu_1\rangle|a,-\vec{k},\nu_2\rangle \\ &= (-)^{2\sigma+2s}|a,-\vec{k},\nu_2\rangle|\bar{a},+\vec{k},\nu_1\rangle \end{aligned} \tag{30}$$

where the second lines have been derived by interchanging two wave functions, which brings in a factor $(-)^{2s}$, positive for mesons and negative for fermions.

The effect of C and G on the two-particle states can now be worked out. By interchanging the subscripts 1 and 2 and by the operation $\vec{k} \to -\vec{k}$, we obtain

$$C|a\bar{a}\nu\rangle = (-)^{\ell+S+\nu}\ |a\bar{a}\ -\nu\rangle \tag{31}$$

$$G|a\bar{a}\nu\rangle = (-)^{\ell+S+I}\ |a\bar{a}\nu\rangle \tag{32}$$

For completeness, we also work out the effect of the parity operation(Π) on the two-particle states. Since antifermions have opposite intrinsic parities to those of their fermion partners, the Π operation brings in the factor $(-)^{2s}$. In addition, the 3-momentum $\vec{k}$ changes sign under the Π operation. Therefore, we have

$$\Pi|a,+\vec{k},\nu_1\rangle|\bar{a},-\vec{k},\nu_2\rangle = (-)^{2s}|a,-\vec{k},\nu_1\rangle|\bar{a},+\vec{k},\nu_2\rangle \tag{33}$$

So, again by using the operation $\vec{k} \to -\vec{k}$, we obtain the familiar result

$$\Pi|a\bar{a}\nu\rangle = (-)^{\ell+2s}\ |a\bar{a}\nu\rangle \tag{34}$$

It follows from (31) that a particle-antiparticle with $\nu = 0$ is in an eigenstate of C with its eigenvalue $(-)^{\ell+S}$. This result applies to all neutral $N\bar{N}$, $q\bar{q}$, $K\bar{K}$ and $\pi\pi$ systems, with $S = 0$ for dikaon and dipion systems. For all ν, a particle-antiparticle system has the G-parity equal to $(-)^{\ell+S+I}$ [see(32)]. Charged $N\bar{N}$, $q\bar{q}$, $K\bar{K}$ systems have $I = 1$, so that their G-parity is $(-)^{\ell+S+1}$ (again $S = 0$ for dikaons). Since the G-parity is $+1$ for dipions, one has $\ell + I =$ even for any $\pi\pi$ system. For all ν, the intrinsic parity of a particle-antiparticle system is given by $(-)^{\ell+2s}$.

5 Three-Particle State: A $K\bar{K}\pi$ System

As an example of the treatment required for three-particle states, we shall work out the case of a neutral $K\bar{K}\pi$ system, which may be in an isospin state of either 0 or 1. Independent of the I-spin, we will show that a wave function for the neutral $K\bar{K}\pi$ system can be devised such that it is in an eigenstate of C with its eigenvalue ± 1 for the state $(K\bar{K})^0\pi^0$ but that it is in an eigenstate of G with its eigenvalue of ± 1 for the state $(K\bar{K})^{\pm}\pi^{\mp}$. In addition, we will also work out in detail the wave functions which result in the case of an intermediate state $K^*(890)$ or $K^*(1420)$. For brevity of notation, we forego the use of ket states used in previous sections; whenever possible, we use the particle names themselves to represent their wave functions.

Consider an arbitrary amplitude A for the state $K\bar{K}\pi$ and expand it in terms of the orbital angular-momentum states for the $K\bar{K}$ subsystem:

$$A = \sum_{\ell} a_{\ell}\psi_{\ell}(K\bar{K}) \tag{35}$$

where the argument denotes ordering of particle momenta in the sense that K has $+\vec{k}$ and $\vec{K}$ has $-\vec{k}$ in the $K\bar{K}$ CM system. Denote by B the amplitude resulting from A by interchanging K and $\bar{K}$. Then, we find

$$\begin{aligned} B &= \sum_{\ell} a_{\ell}\psi_{\ell}(\bar{K}K) \\ &= \sum_{\ell} (-)^{\ell} a_{\ell}\psi_{\ell}(K\bar{K}) \end{aligned} \tag{36}$$

where the second line derives from the property of the spherical harmonics [see(28)]. We can now construct two orthogonal wave functions:

$$\Psi_+ = \frac{1}{2}(A + B) = \sum_{\ell=\text{even}} a_{\ell}\psi_{\ell}(K\bar{K}) \tag{37}$$

$$\Psi_- = \frac{1}{2}(A - B) \quad = \quad \sum_{\ell=\mathrm{odd}} a_\ell \psi_\ell (K\bar{K}) \tag{38}$$

A neutral $K\bar{K}\pi$ system comes in two varieties: $(K\bar{K})^0\pi^0$ and $(K\bar{K})^\pm\pi^\mp$. From the previous section, we know that a $(K\bar{K})^0$ system has the C-parity $(-)^\ell$ and a π^0 has the positive C-parity, so that

$$C\Psi_\pm = \pm\Psi_\pm \qquad \text{for} \quad (\mathrm{K\bar{K}})^0\pi^0 \tag{39}$$

This shows that a neutral $K\bar{K}\pi$ state with a π^0 has the C-parity $+1$ if its wave function is even under K and $\bar{K}$ interchange, and the C-parity is -1 if it is odd under the interchange. A charged $K\bar{K}$ system has necessarily $I = 1$. Therefore, the G-parity for $(K\bar{K})^\pm$ is $(-)^{\ell+1}$. But then π has an odd G- parity, so that

$$G\Psi_\pm = \pm\Psi_\pm \quad \text{for} \quad (\mathrm{K\bar{K}})^\pm\pi^\mp \tag{40}$$

In other words, a neutral $K\bar{K}\pi$ state with a charged π has the G-parity $+1(-1)$ if the wave function is even(odd) under the K and $\bar{K}$ interchange.

Let us recapitulate the results so far for a neutral $K\bar{K}\pi$ system in a given eigenstate of I, C and G. Regardless of the I-spin for the three-body system, its C-parity is even(odd) if the $K\bar{K}$ pair has even(odd) orbital angular momenta for the state $(K\bar{K})^0\pi^0$, whereas its G-parity is even(odd) if the $K\bar{K}$ pair has even(odd) angular momenta for the state $(K\bar{K})^\pm\pi^\mp$. It is worth emphasizing that, although π^0 is also a G-eigenstate, the G-parity for $(K\bar{K})^0\pi^0$ is not known until the I-spin for the three-body state is determined, since a neutral $K\bar{K}$ system can come with its isospin either 0 or 1. Note that these results hold whatever sequential decay the $K\bar{K}\pi$ system may undergo. In particular, a $K\pi$ intermediate state can be either $J^{PC} = 1^{--}$ $K^*(890)$ or $J^{PC} = 2^{++}$ $K^*(1420)$. Therefore, C or G eigenstates depend on the properties of K and $\bar{K}$ interchange and *not* on the intrinsic parities of the nonets to which $K\pi$ or $\bar{K}\pi$ may belong.

We now turn to the problem of constructing a complete wave function for $K\bar{K}\pi$ with an intermediate $I = 1/2$ K^*. We first start by writing down the two K^* states with positive strangeness:

$$\begin{aligned} K^{*+} &= \sqrt{\frac{2}{3}}\pi^+K^0 - \sqrt{\frac{1}{3}}\pi^0K^+ \\ K^{*0} &= \sqrt{\frac{1}{3}}\pi^0K^0 - \sqrt{\frac{2}{3}}\pi^-K^+ \end{aligned} \tag{41}$$

A neutral $K^*\bar{K}$ system in an isospin I can be written

$$A_I = \sqrt{\frac{1}{2}}[K^{*+}K^- - (-)^I K^{*0}\bar{K}^0] \tag{42}$$

where I=0 or 1. Combining (41) and (42), we obtain

$$\begin{aligned} A_I = &\sqrt{\frac{1}{3}}[(\pi^+K^0)K^- + (-)^I(\pi^-K^+)\bar{K}^0] \\ &-\sqrt{\frac{1}{6}}[(\pi^0K^+)K^- + (-)^I(\pi^0K^0)\bar{K}^0] \end{aligned} \tag{43}$$

where parentheses are used to denote K^* states and the ordering of particle names indicates different momenta, e.g. $\vec{p}_1, \vec{p}_2$ and $\vec{p}_3$ in the three-body CM system. Let us denote by B_I the wave function which is obtained from A_I by K and $\bar{K}$ interchange.

We can then define

$$\Phi_g = \sqrt{\frac{1}{2}}[A_I + gB_I] = \Gamma_g + \Theta_g \tag{44}$$

where $g = \pm 1$. Collecting $\pi^{\pm}$'s into Γ_g and π^0's into Θ_g, one finds

$$\begin{aligned} \Gamma_g \;\; = \;\; &\sqrt{\frac{1}{6}}\Big\{[(\pi^+K^0)K^- + g(\pi^+K^-)K^0] \\ &\qquad + (-)^I[(\pi^-K^+)\bar{K}^0 + g(\pi^-\bar{K}^0)K^+]\Big\} \end{aligned} \tag{45}$$

$$\begin{aligned} \Theta_g \;\; = \;\; &-\sqrt{\frac{1}{12}}\Big\{[(\pi^0K^+)K^- + g\,(\pi^0K^-)K^+] \\ &\qquad + (-)^I\,[(\pi^0K^0)\bar{K}^0 + g\,(\pi^0\bar{K}^0)K^0]\Big\} \end{aligned} \tag{46}$$

In order to gain insight to the above formula, it is helpful to define $K\bar{K}$ amd $\bar{K}K$ states in isospin σ,

$$F_\sigma = \sqrt{\frac{1}{2}}[K^+K^- - (-)^\sigma K^0\bar{K}^0] \tag{47}$$

$$\bar{F}_\sigma = \sqrt{\frac{1}{2}}[\bar{K}^0K^0 - (-)^\sigma K^-K^+] \tag{48}$$

These formulae show that neutral $K\bar{K}$ states of (46) have isospins $\sigma = 1$ if $I = 0$ and $\sigma = 0$ if $I = 1$. We conclude, therefore, under the G operation,

$$G\,\Gamma_g = g\Gamma_g \tag{49}$$

$$G\,\Theta_g = g(-)^I\Theta_g \tag{50}$$

$$C\,\Theta_g = g\Theta_g \tag{51}$$

We find once more that a neutral $K\bar{K}\pi$ system with a charged π has a G-parity g, whereas its G-parity is $g(-)^I$ if it has a π^0. In other words, it is the C-parity which is independent of I for the $(K\bar{K})^0\pi^0$ system. From (49) and (50), it is clear that g takes on a different meaning depending on the functions; this is a consequence of the fact that one is exploring the properties of a wave function symmetrized through interchange of K and $\bar{K}$. One may, instead, construct an eigenstate of G-parity alone, which leads to a wave function different from (44). For a thorough analysis of this and other related topics, the reader is referred to another note[1].

It is frequently the case in experiments that both charge states $(K\bar{K})^+\pi^-$ and $(K\bar{K})^-\pi^+$ are measured. Each event of course comes in either one or the other charge state, so that the two charge states cannot interfere with each other. Assuming that the events are dominated either by $I = 0$ states or by $I = 1$ states but not both[1], one may combine the two data sets in two different ways. Since the three variables describing orientation of the three-particle system can be defined equally for the two data sample, one needs to investigate only the two Dalitz-plot variables. For the purpose,let us define the squares of $K\pi$ effective masses,

$$\begin{aligned} s_1 &= M^2(\pi^+K^0), \qquad s_2 = M^2(\pi^+K^-) \\ s_3 &= M^2(\pi^-K^+), \qquad s_4 = M^2(\pi^-\bar{K}^0) \end{aligned} \tag{52}$$

Now we need to write down the general amplitude for the two data sample separately, from (45),

$$\phi_g^+ = (\pi^+K^0)K^- + g(\pi^+K^-)K^0 \tag{53}$$

$$\phi_g^- = (\pi^-K^+)\bar{K}^0 + g(\pi^-\bar{K}^0)K^+ \tag{54}$$

where the superscripts designate π^+ and π^- charge states. We can assume, quite generally, that both charge states produce the same admixture of G-parity eigenstates:

$$\phi^+ = a\phi_+^+ + b\phi_-^+ \tag{55}$$

$$\phi^- = a\phi_+^- + b\phi_-^- \tag{56}$$

where a and b are two arbitrary complex numbers.

In combining the two data, it is natural to respect the strangeness and equate the Dalitz-plot variables as $s_1 = s_3$ and $s_2 = s_4$. An examination of (53) and (54) shows that this amounts to equating the amplitudes as follows:

$$\begin{aligned} D &= (\pi^+K^0)K^- = (\pi^-K^+)\bar{K}^0 \\ E &= (\pi^+K^-)K^0 = (\pi^-\bar{K}^0)K^+ \end{aligned} \tag{57}$$

Submitting these into (55) and (56), one finds

$$\phi^+ = a(D+E) + b(D-E) \tag{58}$$

$$\phi^- = a(D+E) + b(D-E) \tag{59}$$

One has an option of combining $K\pi$ variables according to their net charge i.e. neutral or charged. In that case, one then sets $s_1 = s_4$ and $s_2 = s_3$. The amplitudes are now to be combined in the following way:

$$D = (\pi^+ K^0)K^- = (\pi^- \bar{K}^0)K^+ \tag{60}$$

$$E = (\pi^+ K^-)K^0 = (\pi^- K^+)\bar{K}^0$$

These lead to the overall amplitudes

$$\phi^+ = a(D+E) + b(D-E) \tag{61}$$

$$\phi^- = a(D+E) - b(D-E) \tag{62}$$

The difference between the two approaches is now apparent; it the minus sign in the second term in (62) which is different from (59). Note that, when one combines two charge samples, those with a π^+ and with a π^-, one is in fact combining the squares of the amplitudes

$$|\phi|^2 = |\phi^+|^2 + |\phi^-|^2 \tag{63}$$

This shows that, in the latter approach in which neutral and charged $K\pi$ variables are used, one is cancelling out the G-parity plus-and-minus interference terms(the terms with a*b). On the other hand, in the former approach in which the $K\pi$ variables are grouped together according their strangeness, one is in fact reinforcing the interference term. The reason for this difference can be traced to the fact that the strangeness is a conserved quantum number in strong interactions. It should be emphasized that, in a spin-parity analysis, presence of two G-parity states is detected most sensitively through their interference term.

We end this section by commenting on the branching ratios one expects with a K^* intermediate state. For the purpose, it is best to go back to (43). We see that $BR = 1/3$ for both $(\pi^+ K^0)K^-$ and $(\pi^- K^+)\bar{K}^0$ whereas $BR = 1/6$ for $(\pi^0 K^+)K^-$ and $(\pi^0 K^0)\bar{K}^0$. If neutral K states are detected through $K_S \to \pi^+\pi^-$, one needs to augment the branching ratios by an additional factor $\approx 1/3$ for each K^0 and/or $\bar{K}^0$ appearing in each $K\bar{K}\pi$ state. Note that these branching ratios are independent of the total I-spin for the three-body system.

References

[1] S. U. Chung, 'Analysis of $K\bar{K}\pi$ systems (Version I),' BNL-QGS-98-901