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Unitarized Breit-Wigner Form for Two Overlapping Resonances

—Version III—

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Abstract

A generalization of the Breit-Wigner form for two nearby resonances, which guarantees unitarity, is described.

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1 Two resonances in a single channel

Consider a problem of describing two overlapping resonances coupling to a single channel consisting of two pseudoscalars, in which unitarity is imposed. The standard technique is to use the K-matrix formalism[1].

Let m_a and m_b be the masses of two resonances in an ℓ-wave.

$$K = K_a + K_b = \frac{m_a \Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b \Gamma_b(m)}{m_b^2 - m^2} \tag{1}$$

i.e. the resonances are summed in the K-matrix. The mass-dependent widths are given by

$$\begin{aligned} \Gamma_a(m) &= \Gamma_a^0 \left(\frac{m_a}{m}\right) \left(\frac{q}{q_a}\right) \left(\frac{q}{q_a}\right)^{2\ell} \\ \Gamma_b(m) &= \Gamma_b^0 \left(\frac{m_a}{m}\right) \left(\frac{q}{q_b}\right) \left(\frac{q}{q_b}\right)^{2\ell} \end{aligned} \tag{2}$$

where q is the breakup momentum of a state of mass m into two pseudoscalars and Γ_a^0 and Γ_b^0 are two widths of the resonances when they are far apart, as shown below. It is seen that K is real, and this ensures unitarity. In the P-vector approach, the general invariant production amplitude for the two overlapping resonances is given by

$$F = \frac{P_a + P_b}{1 - i\,(K_a + K_b)} \tag{3}$$

where

$$\begin{aligned} P_a &= \beta_a \, \frac{m_a \Gamma_a^0}{m_a^2 - m^2} \left(\frac{q}{q_a}\right)^{\ell} \\ P_b &= \beta_b \, \frac{m_b \Gamma_b^0}{m_b^2 - m^2} \left(\frac{q}{q_b}\right)^{\ell} \end{aligned} \tag{4}$$

β_a and β_b are unitless complex constants specifying the productions strengths of the states a and b.

Now assume that the two states are far apart, compared to their widths. Then, if m is either close to m_a or to m_b, one could write

$$\begin{aligned} F \simeq \beta_a \, &\frac{m_a \Gamma_a^0}{m_a^2 - m^2 - i\, m_a \Gamma_a(m)} \left(\frac{q}{q_a}\right)^{\ell} \\ +\beta_b \, &\frac{m_b \Gamma_b^0}{m_b^2 - m^2 - i\, m_b \Gamma_b(m)} \left(\frac{q}{q_b}\right)^{\ell} \end{aligned} \tag{5}$$

This shows that Γ_a^0 and Γ_b^0 are the natural widths of the resonances when they are far apart.

Consider next the case in which the masses are the same, i.e. $m_0 = m_a = m_b$ but their widths are different. Then one finds

$$F = \frac{m_0\,(\beta_a \Gamma_a^0 + \beta_b \Gamma_b^0)}{m_0^2 - m^2 - i\,m_0\,[\Gamma_a(m) + \Gamma_b(m)]} \left(\frac{q}{q_0}\right)^{\ell} \tag{6}$$

This shows that the effective width is the sum of the two initial widths.

It is important to recognize that very few examples of practical importance are reducible to a single-channel problem. Consider as an example the case of $a = f_2(1270)$ and $b = f_2'(1525)$ decaying into the channel $\eta\eta$. This is clearly a three-channel problem, consisting of the cannels $\pi\pi$, $K\bar{K}$ and $\eta\eta$. One could, nevertheless, approximate the problem with a single-channel case by taking only the dominant decay modes in the denominator. Thus, one could take the mass-dependent width $\Gamma_a(m)$ to correspond to the decay $f_2(1270) \to \pi\pi$ and take $\Gamma_b(m)$ to correspond to $f_2'(1525) \to K\bar{K}$. But the barrier factors q^ℓ $(\ell = 2)$ which appear in P_a and P_b should refer to the decay channel $\eta\eta$.

In practice, the barrier factors q^ℓ are often replaced by the Blatt-Weisskopf functions. The reader may consult a previous note by the author[2] for these functions up to $\ell = 4$.

2 Two resonances in three channels

To be specific, consider two states $a = f_2(1270)$ and $b = f_2'(1525)$ decaying into channels $1 = \pi\pi$, $2 = K\bar{K}$ and $3 = \eta\eta$. The branching ratios are given in Table I.

Table I. Branching Ratios(%)

		Decay Modes		
$I^G\ J^{PC}$	State	$\pi\pi$	$K\bar{K}$	$\eta\eta$
$0^+\ 2^{++}$	$f_2(1270)$	$84.7^{+2.6}_{-1.2}$	4.6 ± 0.5	$(4.5 \pm 1.0) \times 10^{-3}$
$0^+\ 2^{++}$	$f_2'(1525)$	$(8.2 \pm 1.5) \times 10^{-3}$	88.8 ± 3.1	10.3 ± 3.1

We shall assume that the channels 1, 2 and 3 are produced by pion exchange in the reaction $\pi^- p \to f_2(\text{or } f_2')\, n$. In other words, we consider the reactions $\pi\pi \to \pi\pi$, $\pi\pi \to K\bar{K}$ and $\pi\pi \to \eta\eta$. This problem is described in general by 3-dimensional Lorentz-invariant $\widehat{K}$ and $\widehat{T}$ matrices, as follows:

$$\widehat{T} = (I - i\,\widehat{K}\,\rho)^{-1}\,\widehat{K} \tag{7}$$

The elements of the $\widehat{K}$ matrix are given by

$$\begin{aligned}
\widehat{K}_{11}\,\rho_1 &= \gamma_{a1}^2\,\Delta_a(m)\,B_{a1}^2\;\rho_{a1}(m) + \gamma_{b1}^2\,\Delta_b(m)\,B_{b1}^2\;\rho_{b1}(m)\\
\widehat{K}_{22}\,\rho_2 &= \gamma_{a2}^2\,\Delta_a(m)\,B_{a2}^2\;\rho_{a2}(m) + \gamma_{b2}^2\,\Delta_b(m)\,B_{b2}^2\;\rho_{b2}(m)\\
\widehat{K}_{33}\,\rho_3 &= \gamma_{a3}^2\,\Delta_a(m)\,B_{a3}^2\;\rho_{a3}(m) + \gamma_{b3}^2\,\Delta_b(m)\,B_{b3}^2\;\rho_{b3}(m)\\
\widehat{K}_{12}\,\rho_2 &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\,B_{a1}B_{a2}\;\rho_{a2}(m) + \gamma_{b1}\gamma_{b2}\,\Delta_b(m)\,B_{b1}B_{b2}\;\rho_{b2}(m)\\
\widehat{K}_{23}\,\rho_3 &= \gamma_{a2}\gamma_{a3}\,\Delta_a(m)\,B_{a2}B_{a3}\;\rho_{a3}(m) + \gamma_{b2}\gamma_{b3}\,\Delta_b(m)\,B_{b2}B_{b3}\;\rho_{b3}(m)\\
\widehat{K}_{13}\,\rho_3 &= \gamma_{a1}\gamma_{a3}\,\Delta_a(m)\,B_{a1}B_{a3}\;\rho_{a3}(m) + \gamma_{b1}\gamma_{b3}\,\Delta_b(m)\,B_{b1}B_{b3}\;\rho_{b3}(m)
\end{aligned} \tag{8}$$

where $\widehat{K}_{ij} = \widehat{K}_{ji}$ and

$$\begin{aligned}
\Delta_\alpha(m) &= \frac{m_\alpha\,\Gamma_\alpha^0}{m_\alpha^2 - m^2}\\
B_{\alpha\,i} &= \left(\frac{q_i}{q_{\alpha\,i}}\right)^2\\
\rho_{\alpha\,i} &= \left(\frac{m_\alpha}{m}\right)\,\left(\frac{q_i}{q_{\alpha\,i}}\right)
\end{aligned} \tag{9}$$

α stands for a or b, and i is the channel number 1, 2 or 3. q_i is the breakup momentum for channel i, and $q_{\alpha\,i} = q_i\,(m = m_\alpha)$. Finally, $\gamma_{\alpha\,i}^2$ is the branching ratio for a state α decaying into channel i and is real—note that γ_{a3}^2 and γ_{b1}^2 are very small numbers, as seen in Table I.

The formula (7) is strictly applicable only in the limit of the physical processes $i \to j$, where i and j are the channel numbers. In practice, one of the initial particles is a Reggeon, whose mass square ranges over a continuum t, the four-momentum square. One must, therefore, take out the mass dependence of the initial state and write, for the invariant amplitudes F,

$$F = (I - i\,\widehat{K}\,\rho)^{-1}\,R \equiv Q\,R \tag{10}$$

where $\widehat{K}$ is as given in (8) and Q is the inverse of $(I - i\,\widehat{K}\,\rho)$, but now R is the modified form of $\widehat{K}$, such that channel dependence i of R_{fi} is set to 1 except $\gamma_{\alpha\,i}$ and $\rho_{\alpha\,i}$'s are removed everywhere:

$$\begin{aligned}
R_{11} &= \gamma_{a1}^2\,\Delta_a(m)\,B_{a1} + \gamma_{b1}^2\,\Delta_b(m)\,B_{b1}\\
R_{22} &= \gamma_{a2}^2\,\Delta_a(m)\,B_{a2} + \gamma_{b2}^2\,\Delta_b(m)\,B_{b2}\\
R_{33} &= \gamma_{a3}^2\,\Delta_a(m)\,B_{a3} + \gamma_{b3}^2\,\Delta_b(m)\,B_{b3}\\
R_{12} &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\,B_{a1} + \gamma_{b1}\gamma_{b2}\,\Delta_b(m)\,B_{b1}\\
R_{23} &= \gamma_{a2}\gamma_{a3}\,\Delta_a(m)\,B_{a2} + \gamma_{b2}\gamma_{b3}\,\Delta_b(m)\,B_{b2}\\
R_{13} &= \gamma_{a1}\gamma_{a3}\,\Delta_a(m)\,B_{a1} + \gamma_{b1}\gamma_{b3}\,\Delta_b(m)\,B_{b1}
\end{aligned} \tag{11}$$

One has adopted here a convention that, if i and f are initial and final states, then one writes R_{fi}. This formula is reminiscent of the P-vector approach adopted in the previous section, except that $\gamma_{\alpha i}$ is real (but it could be negative). Note that R_{ij} is not a symmetric matrix; R_{21}, R_{32} and R_{31} are obtained by interchanging i and j of R_{ij} given in (11).

Consider now the process $\pi\pi \to \eta\eta$, observed in the reaction $\pi^- p \to \eta\eta\, n$. The appropriate invariant amplitude for this process is, from (10),

$$F_{31} = Q_{31}\, R_{11} + Q_{32}\, R_{21} + Q_{33}\, R_{31} \tag{12}$$

Note that $1 = \pi\pi$ is the initial state and $3 = \eta\eta$ is the final state. One sees that this procedure leads to no *free* parameters, and it may turn out to be too restrictive. But then, if necessary, one can switch over to the P-vector approach and replace $\gamma_{\alpha i}$ in R_{fi} of (11) with an arbitrary complex parameter $\beta_{\alpha i}$. We are now ready to derive an approximate formula, based on the assumptions

$$\begin{aligned} \gamma_{a3} &= \gamma_{b1} = 0 \\ \gamma_{a2}^2 &= \gamma_{b3}^2 = 0 \end{aligned} \tag{13}$$

Note that $\gamma_{a2} \simeq 0.21$ and $\gamma_{b3} \simeq 0.32$ are significant, and they are retained in the approximation. From Table I, one sees that these assumptions amount to a 10% error in general. The modified $\widehat{K}$ and R now assume the forms

$$\begin{aligned} \widehat{K}_{11}\,\rho_1 &= \gamma_{a1}^2\,\Delta_a(m)\,B_{a1}^2\;\rho_{a1}(m) \\ \widehat{K}_{22}\,\rho_2 &= \gamma_{b2}^2\,\Delta_b(m)\,B_{b2}^2\;\rho_{b2}(m) \\ \widehat{K}_{33}\,\rho_3 &= 0 \\ \widehat{K}_{12}\,\rho_2 &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\,B_{a1}B_{a2}\,\rho_{a2}(m) \\ \widehat{K}_{21}\,\rho_1 &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\,B_{a1}B_{a2}\,\rho_{a1}(m) \\ \widehat{K}_{23}\,\rho_3 &= \gamma_{b2}\gamma_{b3}\,\Delta_b(m)\,B_{b2}B_{b3}\,\rho_{b3}(m) \\ \widehat{K}_{32}\,\rho_2 &= \gamma_{b2}\gamma_{b3}\,\Delta_b(m)\,B_{b2}B_{b3}\,\rho_{b2}(m) \\ \widehat{K}_{13}\,\rho_3 &= K_{31}\,\rho_1 = 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} R_{11} &= \gamma_{a1}^2\,\Delta_a(m)\,B_{a1} \\ R_{22} &= \gamma_{b2}^2\,\Delta_b(m)\,B_{b2} \\ R_{33} &= 0 \\ R_{12} &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\,B_{a1} \\ R_{23} &= \gamma_{b2}\gamma_{b3}\,\Delta_b(m)\,B_{b2} \\ R_{21} &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\,B_{a2} \\ R_{32} &= \gamma_{b2}\gamma_{b3}\,\Delta_b(m)\,B_{b3} \\ R_{13} &= R_{31} = 0 \end{aligned} \tag{15}$$

Note also that $\widehat{K}_{12}^2 = \widehat{K}_{23}^2 = 0$ and $\widehat{K}_{21}\,R_{12} = \widehat{K}_{32}\,R_{23} = 0$.

$$F_{31} = \frac{i\,\widehat{K}_{32}\,\rho_2\,R_{21}}{1 - i\,\widehat{K}_{22}\,\rho_2} - \frac{\widehat{K}_{32}\,\rho_2\,\widehat{K}_{21}\,\rho_1\,R_{11}}{(1 - i\,\widehat{K}_{11}\,\rho_1)(1 - i\,\widehat{K}_{22}\,\rho_2)} \tag{16}$$

One sees that this formula is very different from (3) of section 1. The first term could be interpreted as the process corresponding to a production of $K\bar{K}$, via $\pi\pi \to K\bar{K}$, which turns into $\eta\eta$ through mediation by the state b. The second term corresponds to a production of $\pi\pi$, which turns into $K\bar{K}$ through the state a, and then finally into $\eta\eta$ mediated by the state b.

Consider next the process $\pi\pi \to \pi\pi$, observed in the reaction $\pi^- p \to \pi\pi\, n$. The invariant amplitude is, in the same approximation,

$$F_{11} = \frac{R_{11}}{1 - i\,\widehat{K}_{11}\,\rho_1} + \frac{i\,\widehat{K}_{12}\,\rho_2\,R_{21}}{(1 - i\,\widehat{K}_{11}\,\rho_1)(1 - i\,\widehat{K}_{22}\,\rho_2)} \tag{17}$$

The first term corresponds to a production of $\pi\pi$, resonating in the state a. In the absence of other open channels, this term alone would suffice. The second term has no simple explanation; note, however, that both the states a and b are involved in the amplitude and that it depends on the channels $\pi\pi$ and $K\bar{K}$ but *not* on $\eta\eta$.

3 Two resonances in two channels

Consider two S-wave states $a = f_0(1300)$ and $b = f_0(1500)$ decaying into $1 = \pi\pi$ and $2 = \eta\eta$. The goal is to study the process $\pi\pi \to \eta\eta$ observed in the reaction $\pi^- p \to \eta\eta\, n$. The invariant amplitude is again given by (10), but the matrices are 2-dimensional. They are

$$\begin{aligned}
\widehat{K}_{11}\,\rho_1 &= \gamma_{a1}^2\,\Delta_a(m)\,\rho_{a1}(m) + \gamma_{b1}^2\,\Delta_b(m)\,\rho_{b1}(m)\\
\widehat{K}_{22}\,\rho_2 &= \gamma_{a2}^2\,\Delta_a(m)\,\rho_{a2}(m) + \gamma_{b2}^2\,\Delta_b(m)\,\rho_{b2}(m)\\
\widehat{K}_{12}\,\rho_2 &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\;\rho_{a2}(m) + \gamma_{b1}\gamma_{b2}\,\Delta_b(m)\,\rho_{b2}(m)\\
\widehat{K}_{21}\,\rho_1 &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m)\;\rho_{a1}(m) + \gamma_{b1}\gamma_{b2}\,\Delta_b(m)\,\rho_{b1}(m)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
R_{11} &= \gamma_{a1}^2\,\Delta_a(m) + \gamma_{b1}^2\,\Delta_b(m)\\
R_{22} &= \gamma_{a2}^2\,\Delta_a(m) + \gamma_{b2}^2\,\Delta_b(m)\\
R_{12} = R_{21} &= \gamma_{a1}\gamma_{a2}\,\Delta_a(m) + \gamma_{b1}\gamma_{b2}\,\Delta_b(m)
\end{aligned} \tag{19}$$

So the process $\pi\pi \to \eta\eta$ is given by

$$F_{21} = \frac{i\,\widehat{K}_{21}\,\rho_1\,R_{11} + (1 - i\,\widehat{K}_{11}\,\rho_1)\,R_{21}}{(1 - i\,\widehat{K}_{11}\,\rho_1)(1 - i\,\widehat{K}_{22}\,\rho_2) + \widehat{K}_{12}\,\rho_2\,\widehat{K}_{21}\,\rho_1} \tag{20}$$

The branching ratios are not well determined for $f_0(1300)$ and $f_0(1500)$. In this case, the formula above may harbor too many unkowns, to be useful—one may have no recourse but to resort to a single-channel description given in section 1.

It is easy to write down the invariant amplitude for the process $\pi\pi \to \pi\pi$, observed in the reaction $\pi^- p \to \pi\pi\, n$:

$$F_{11} = \frac{(1 - i\,\widehat{K}_{22}\,\rho_2)\,R_{11} + i\,\widehat{K}_{12}\,\rho_2\,R_{21}}{(1 - i\,\widehat{K}_{11}\,\rho_1)(1 - i\,\widehat{K}_{22}\,\rho_2) + \widehat{K}_{12}\,\rho_2\,\widehat{K}_{21}\,\rho_1} \tag{21}$$

Once again, a unitarized description of the process $\pi\pi \to \pi\pi$ calls for a knowledge of the processes $\pi\pi \to \eta\eta$ and $\eta\eta \to \pi\pi$.

References

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[3] Review of Particle Physics (Part I), **54**, 1(1996).